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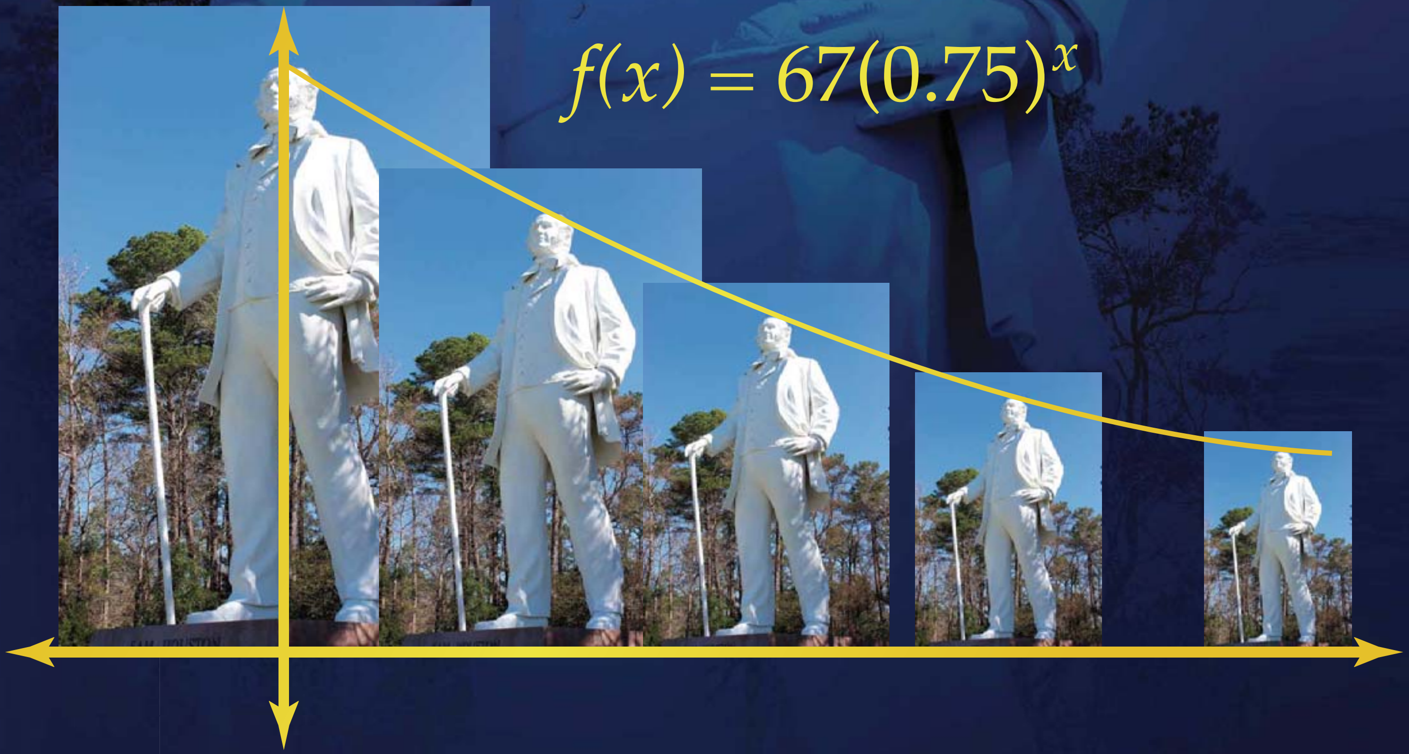

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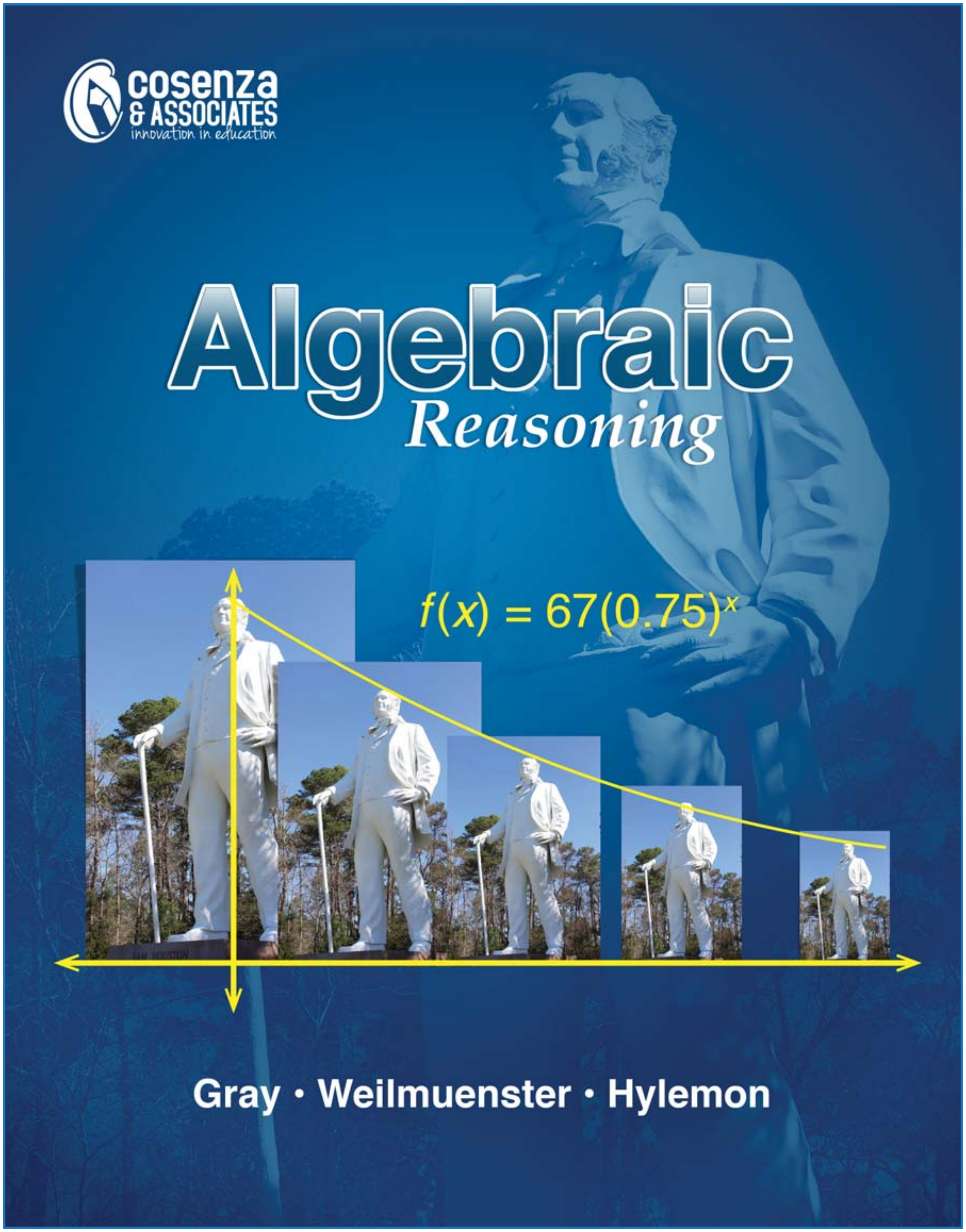

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Algebraic
Reasoning

$f(x) = 67(0.75)^x$

Gray • Weilmuenster • Hylemon

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innovation in education

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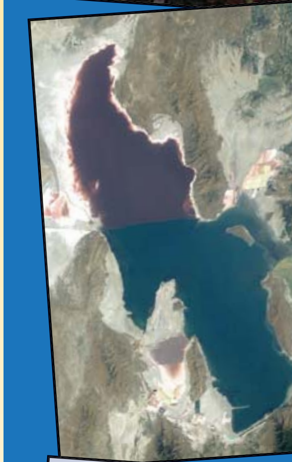

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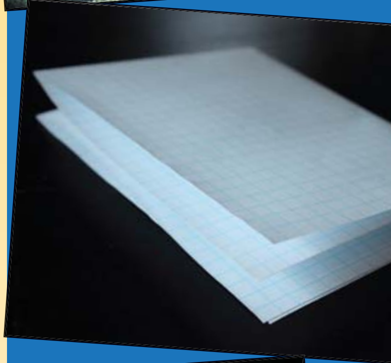
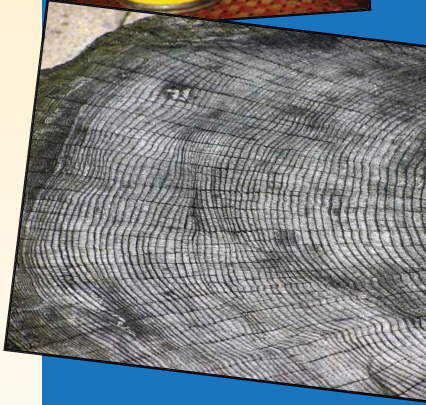
Algebraic Patterns





1.1

Arithmetic and Geometric Sequences

FOCUSING QUESTION How are arithmetic and geometric sequences alike? How are they different?

LEARNING OUTCOMES

- I can determine patterns that identify a linear function or an exponential function.
- I can identify terms of an arithmetic or geometric sequence. (Algebra 1)
- I can write a formula for the n^{th} term of an arithmetic or geometric sequence. (Algebra 1)
- I can use symbols, tables, and language to communicate mathematical ideas.

ENGAGE

Brenda is at the farmer's market. There are several baskets of tomatoes on a table. Each basket contains 6 tomatoes. What sequence would Brenda create if she listed the number of tomatoes in a set of baskets (1 basket, 2 baskets, 3 baskets, etc.)?

EXPLORE

The first few terms of two different sequences are shown.

SEQUENCE 1

SEQUENCE 2

Use toothpicks or counters to build the next two terms of each sequence. Record your information in the table.

SEQUENCE 1	
TERM NUMBER	NUMBER OF TOOTHPICKS
1	
2	
3	
4	
5	
6	
10	

SEQUENCE 2	
TERM NUMBER	NUMBER OF COUNTERS
1	
2	
3	
4	
5	
6	
10	

1. What patterns do you observe in Sequence 1?

2. What relationships do you observe between the term number and the number of toothpicks required to build each term in Sequence 1? *(Hint: Record the multiples of 3 next to the number of toothpicks.)*

3. How does the pattern that you mentioned in question 2 appear in the figures that you constructed?

4. How many toothpicks would you need to build the 10th term of Sequence 1?

5. What patterns do you observe in Sequence 2?

6. What relationships do you observe between the term number and the number of counters required to build each term in Sequence 2? *(Hint: Record powers of 2 next to the number of counters.)*

7. How does the pattern that you mentioned in question 6 appear in the figures that you constructed?

8. How many counters would you need to build the 10th term of Sequence 2?

9. Compare the two sequences. How are they alike? How are they different?

REFLECT

- **Which sequence has a constant difference between the numbers of objects for successive terms? How do you know?**
- **Which sequence has a constant ratio between the numbers of objects for successive terms? How do you know?**

EXPLAIN

A sequence is a set of numbers that are listed in order and that follow a particular pattern. An **arithmetic sequence** is a sequence that has a constant difference between consecutive terms.

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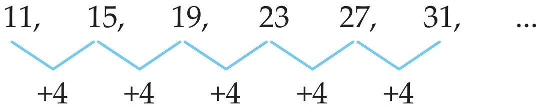
For example, suppose Bernardo earns tickets from the local arcade. He starts out with 11 tickets and earns 4 more tickets for each game he plays. Bernardo can create an arithmetic sequence to show the number of tickets he has after each game.



Notice that the next term in the sequence can be generated by adding 4 to the previous term. You can use recursive notation to generalize an arithmetic sequence like Bernardo's.

$$a_1 = 11$$
$$a_n = a_{n-1} + 4$$

The addend of 4 is used to generate the next term in this arithmetic sequence. In general, the addend in an arithmetic sequence is called a **common difference**, since it is also the difference between consecutive terms.

A **subscript** is used to indicate a special case of a variable. a_1 indicates the first term of a sequence, a_2 indicates the second term of a sequence, and so on. a_1 is read "a-sub one," where "sub" indicates a subscript.

A **geometric sequence** is a sequence that has a constant ratio between consecutive terms. For example, Kayla earns \$0.02 the first week for her allowance, but each week she earns twice as much as she did the week before. She can create a geometric sequence to show the amount of money she earns each week through her allowance.

\$0.02 \$0.04 \$0.08 \$0.16 \$0.32 \$0.64 \$1.28 \$2.56 \$5.12 \$10.24 ...

×2 ×2 ×2 ×2 ×2 ×2 ×2 ×2 ×2

The multiplier of 2 is used to generate the next term in this geometric sequence. In general, the multiplier in a geometric sequence is called a **common ratio**, since it is also the ratio between consecutive terms.

Kayla's sequence can be represented by a recursive rule.

$$a_1 = 0.02$$
$$a_n = 2a_{n-1}$$

ARITHMETIC AND GEOMETRIC SEQUENCES

An arithmetic sequence has a constant addend or common difference. It is an additive relationship between terms of the sequence.

A geometric sequence has a constant multiplier or common ratio. It is a multiplicative relationship between terms of the sequence.

If you have a sequence, you can use the constant difference or constant multiplier to determine subsequent terms of the sequence.

EXAMPLE 1

Anthony worked all summer to save \$1950 for spending money during the school year. He plans to withdraw the same amount from his savings account at the end of each week. Anthony can create an arithmetic sequence that shows the balance of his savings account at the beginning of each week of the school year.

\$1950, \$1885, \$1820

How much money will be in Anthony's savings account at the beginning of the fourth week of the school year? How much money will be in Anthony's savings account at the beginning of the fifth week of the school year?

STEP 1 First, use the existing data to determine the common difference.

TIME (WEEKS)	SAVINGS BALANCE (DOLLARS)
1	$1950
2	$1885
3	$1820

$-\$65 = \$1885 - \$1950$

$-\$65 = \$1820 - \$1855$

The common difference is –$65.

STEP 2 Next, apply the common difference to the balance at the beginning of the third week to determine the balance at the beginning of the fourth week.

$$\$1820 + (-\$65) = \$1755$$

STEP 3 Then, apply the common difference to determine the balance in Anthony's savings account at the beginning of the fifth week of school.

$$\$1755 + (-\$65) = \$1690$$

The balance in Anthony's savings account is $1755 at the beginning of the fourth week of school and $1690 at the beginning of the fifth week of school.

YOU TRY IT! #1

Rachel helps the student council create a paper chain that contains students' written pledges not to bully or tolerate bullying. When Rachel begins stapling, there are 54 inches of paper chain. She measures after adding each link and records the results in a table.

NUMBER OF LINKS RACHEL ADDED	1	2	3
LENGTH OF PAPER CHAIN (INCHES)	$58\frac{1}{2}$	63	$67\frac{1}{2}$

What is the common difference in this situation, and how long will the paper chain be after Rachel has added a total of six links to it?

EXAMPLE 2

At the beginning of the year, an investor puts $1000 into a fund that pays 20% annually. The investor projects how much will be in the fund at the end of each year for the next three years.

$1200, $1440, $1728

How much money will be in the investment fund at the end of four years? How much money will be in the investment fund at the end of five years?

STEP 1 **First, determine the common ratio in this situation.**

$1440 ÷ $1200 = 1.2
$1728 ÷ $1440 = 1.2
The common ratio is 1.2.

STEP 2 **Next, multiply the third value in the geometric sequence by the common ratio to determine the fourth value in the geometric sequence.**

($1728)(1.2) = $2073.60

STEP 3 **Then, multiply by the common ratio to determine the fifth value in the geometric sequence.**

($2073.60)(1.2) = $2488.32

There will be $2073.60 in the fund after four years and $2488.32 in the fund after five years.

YOU TRY IT! #2

A patient takes 500 milligrams of medicine. A nurse charts the amount of medication in the patient's system.

TIME SINCE DOSAGE (HOURS)	MEDICINE IN PATIENT'S SYSTEM (MG)
1	400
2	320
3	256
4	204.8

What is the common ratio in this situation and approximately how much medicine, to the nearest milligram, will remain in the patient's system after five hours?

EXAMPLE 3

For the sequence shown, write a recursive rule and an explicit rule.

4, 6.5, 9, 11.5, 14, …

A **recursive rule** shows how to determine the *n*th term, a_n, using the value of the previous term. An **explicit rule**, like a function, shows how to determine the *n*th term, a_n, using the term number, *n*.

STEP 1 First, determine the common difference or ratio in this situation.

4, 6.5, 9, 11.5, 14, ...

+2.5 +2.5 +2.5 +2.5

The common difference is +2.5.

STEP 2 Next, write the first term in the sequence, a_1. Use the common difference to write a recursive rule relating an to the previous term, a_{n-1}.

$$a_1 = 4$$
$$a_n = a_{n-1} + 2.5$$

STEP 3 Use the common difference to work backwards to determine the value of term 0.

$$\underbrace{1.5, \quad 4,}_{+2.5} \quad 6.5, \quad 11.5, \quad 14, \quad \ldots$$

STEP 4 Use the common difference and the value of term 0 to write an explicit rule with term 0 as the starting point and the common difference as the rate of change.

$$a_n = 1.5 + 2.5n$$

YOU TRY IT! #3

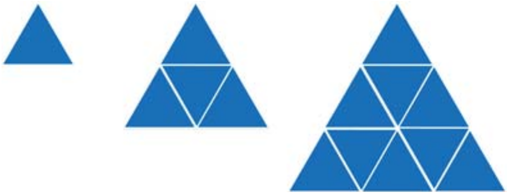
Write a recursive rule and an explicit rule for the sequence $6, 7\frac{1}{3}, 8\frac{2}{3}, 10, 11\frac{1}{3}, \ldots$

PRACTICE/HOMEWORK

For questions 1 – 4 write an explicit rule that describes the number of items used to construct the pattern in terms of the term number, n.

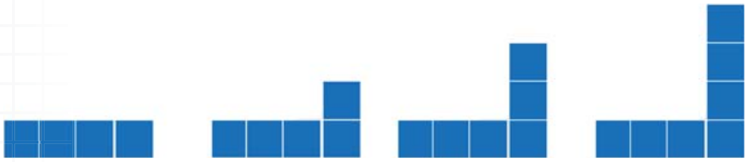
1.

2.



3.

4.



For questions 5 and 6, use the following situation.

FINANCE

Segway Tours in Corpus Christi charges $12 an hour to rent a Segway and an additional fee of $4 for the required helmet. David can create an arithmetic sequence that shows the cost of renting a Segway.

16, 28, 40, 52, …

5. How much will David spend to rent the Segway with a helmet for 6 hours?

6. Write a function rule that describes the cost of renting a Segway, $f(n)$, in terms of the number of hours, n, David rents the Segway.

For questions 7 and 8, use the following situation.

SCIENCE

Roger dropped a ball from a height of 1000 centimeters. The height of the ball is 80% of the previous height after each bounce of the ball. Roger can create a geometric sequence that shows the height of the ball at the end of each bounce.

800, 640, 512, 409.6, …

7. What is the height of the ball after the 5th bounce?

8. Write a function rule that describes the height of the ball, $f(n)$, after the number of bounces, n, the ball makes.

For questions 9 and 10, use the following situation.

FINANCE

Clayton opens a savings account with \$11 he got from his grandmother. Each month after the initial deposit, he adds \$15 to the account. Clayton can create an arithmetic sequence that shows the balance of his savings account at the end of each month after he deposits funds in the savings account.

$$26, 41, 56, \ldots$$

9. How much money will Clayton have in his account after he deposits money for 12 months?

10. Write an explicit rule that describes the amount of money in Clayton's account, a_n, in terms of the number of months, n, he deposits money.

For questions 11 – 16 determine whether the sequences shown are arithmetic or geometric sequences. Then, write a recursive rule and an explicit rule.

11. $1, 8, 15, 22, 29, \ldots$

12. $2, 6, 18, 54, 162, \ldots$

13. $-10, -6.5, -3, 0.5, 4, 7.5, \ldots$

14. $1.5, 7.5, 37.5, 187.5 \ldots$

15. $64, 16, 4, 1, 0.25, \ldots$

16. $147, 127, 107, 87, 67, \ldots$

For questions 17 – 20 for each recursive rule and explicit rule given below, write the first 4 terms in the sequence.

17. $a_1 = 9.5; a_n = a_{n-1} + 6.5$
$a_n = 3 + 6.5n$

18. $a_1 = 3; a_n = 4a_{n-1}$
$a_n = 3(4)^{n-1}$

19. $a_1 = 625; \mathrm{a}_n = \mathrm{a}_{n-1} \div 5 = \frac{1}{5}a_{n-1}$

$a_n = 625\left(\frac{1}{5}\right)^{n-1}$

20. $a_1 = 140; a_n = a_{n-1} - 30$
$a_n = 170 - 30n$

1.2 Writing Linear Functions

FOCUSING QUESTION What are the characteristics of a linear function?

LEARNING OUTCOMES

- I can determine patterns that identify a linear function from its related finite differences.
- I can determine the linear function from a table using finite differences, including any restrictions on the domain and range.
- I can analyze patterns to connect the table to a function rule and communicate the linear pattern as a function rule.

ENGAGE

Marcus works in an orchard. Each row in the orchard contains 20 trees. How can Marcus use this information to make a table of values to represent the number of trees in the orchard?

EXPLORE

Miranda and her family will spend their summer vacation on the beach. They plan to rent a beach house that has a fixed cleaning fee and a daily rental fee. The table below shows the rental cost for each of a certain number of days.

NUMBER OF DAYS	RENTAL COST
1	$170
2	$285
3	$400
4	$515
5	$630
6	$745
7	$860

1. What is the difference between the numbers of days in consecutive rows in the table?

2. What is the difference between the rental cost in consecutive rows in the table?

3. Use the pattern in the table to predict the rental cost for 0 days.

4. Based on the pattern in the table, what do you think the cleaning fee is? Explain how you know.

5. Based on the pattern in the table, what do you think the daily rental fee is? Explain how you know.

6. Use the pattern in the table to write an equation that shows the relationship between n, the number of days the beach house will be rented and r, the total rental cost.

Miranda and her sister have pooled their money for meals. From the initial amount of money they placed in an envelope, they will spend a certain amount each day on food. The table below shows the balance of money remaining in the envelope after a certain number of days.

NUMBER OF DAYS	BALANCE
1	$225
2	$190
3	$155
4	$120
5	$85
6	$50
7	$15

7. What is the difference between the numbers of days in consecutive rows in the table?

8. What is the difference between the balances in consecutive rows in the table?

9. Use the patterns in the table to predict the balance on day 0.

10. Based on the patterns in the table, how much money do you think Miranda and her sister initially pooled? Explain how you know.

11. Based on the patterns in the table, how much money did Miranda and her sister spend on meals each day? Explain how you know.

12. Use the patterns in the table to write an equation that shows the relationship between n, the number of days of the vacation and b, the balance of pooled money remaining.

REFLECT

- **What do you notice about the differences in values for successive table rows for both the independent and dependent variable?**
- **What relationship exists between the ratio of the differences in the dependent variable to the differences in the independent variable and the equations that you have written?**

EXPLAIN

The differences in values for successive table rows are called **finite differences**. When you have a table of data, you can use finite differences to determine the type of function the data represents and to write a function representing the relationship between the variables in the table.

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Let's look more closely at a linear function. The table on the next page shows the relationship between x and $f(x)$. In a linear function, $f(x) = mx + b$, m represents the slope or rate of change, and b represents the y-coordinate of the y-intercept, or starting point.

The slope, which is the rate of change, of a linear function connecting two points, (x_1, y_1) and (x_2, y_2) is found using the slope formula.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

x	$y = f(x)$
0	b
1	$b + m$
2	$b + 2m$
3	$b + 3m$
4	$b + 4m$
5	$b + 5m$

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta y = (b + m) - b = b + m - b = b - b + m = m$

$\Delta y = (b + 2m) - (b + m) = b + 2m - b - m = b - b + 2m - m = m$

$\Delta y = (b + 3m) - (b + 2m) = b + 3m - b - 2m = b - b + 3m - 2m = m$

$\Delta y = (b + 4m) - (b + 3m) = b + 4m - b - 3m = b - b + 4m - 3m = m$

$\Delta y = (b + 5m) - (b + 4m) = b + 5m - b - 4m = b - b + 5m - 4m = m$

Notice that in the table, the difference between each pair of x-values, Δx, is 1 and the difference between each pair of y-values is m. The finite differences in y-values, Δy, for a linear function are the same, so we can say that the finite differences are constant.

FINITE DIFFERENCES AND LINEAR FUNCTIONS

In a linear function, the finite differences between successive y-values, Δy, are constant if the differences between successive x-values, Δx, are also constant.

If the finite differences in a table of values are constant, then the values represent a linear function.

You can also use the finite differences to write a linear function describing the relationship between the independent and dependent variables.

EXAMPLE 1

Does the set of data shown below represent a linear function? Justify your answer.

x	y
0	11
1	17
2	23
3	29
4	35

STEP 1 Determine the finite differences between successive x-values and successive y-values.

x	y
0	11
1	17
2	23
3	29
4	35

$\Delta x = 1 - 0 = 1$ $\Delta y = 17 - 11 = 6$

$\Delta x = 2 - 1 = 1$ $\Delta y = 23 - 17 = 6$

$\Delta x = 3 - 2 = 1$ $\Delta y = 29 - 23 = 6$

$\Delta x = 4 - 3 = 1$ $\Delta y = 35 - 29 = 6$

STEP 2 Determine whether or not the ratios of the differences are constant.

The differences in x, Δx, are all 1, so they are constant.

The differences in y, Δy, are all 6, so they are constant.

$\frac{\Delta y}{\Delta x} = \frac{6}{1} = 6$ for all pairs of Δx and Δy.

STEP 3 Determine whether or not the set of data represents a linear function.

Yes, the set of data represents a linear function because the finite differences in y-values are constant when the finite differences in x-values are also constant.

YOU TRY IT! #1

Does the set of data shown below represent a linear function? Justify your answer.

x	y
0	6.4
1	7.2
2	8.0
3	8.8
4	9.6

EXAMPLE 2

Does the set of data shown below represent a linear function? Justify your answer.

x	y
0	17
1	15.5
2	14
3	11.5
4	10

STEP 1 **Determine the first differences between successive x-values and successive y-values.**

x	y
0	17
1	15.5
2	14
3	11.5
4	10

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta y = 15.5 - 17 = -1.5$

$\Delta y = 14 - 15.5 = -1.5$

$\Delta y = 11.5 - 14 = -2.5$

$\Delta y = 10 - 11.5 = -1.5$

STEP 2 **Determine whether or not the differences are constant.**

The differences in x, Δx, are all 1, so they are constant.

The differences in y, Δy, are not all the same, so they are not constant.

STEP 3 **Determine whether or not the set of data represents a linear function.**

No, the set of data does not represent a linear function because the finite differences for the y-values are not constant when the finite differences for the x-values are constant.

YOU TRY IT! #2

Does the set of data shown below represent a linear function? Justify your answer.

x	y
0	7.1
1	7.5
2	8.1
3	8.9
4	9.9

EXAMPLE 3

For the data set below, determine if the relationship is a linear function. If so, determine a function relating the variables.

x	y
1	7.5
2	10
3	12.5
4	15
5	17.5

STEP 1 **Determine the finite differences between successive x-values and successive y-values.**

	x	y	
	1	7.5	
$\Delta x = 2 - 1 = 1$			$\Delta y = 10 - 7.5 = 2.5$
	2	10	
$\Delta x = 3 - 2 = 1$			$\Delta y = 12.5 - 10 = 2.5$
	3	12.5	
$\Delta x = 4 - 3 = 1$			$\Delta y = 15 - 12.5 = 2.5$
	4	15	
$\Delta x = 5 - 4 = 1$			$\Delta y = 17.5 - 15 = 2.5$
	5	17.5	

STEP 2 **Determine whether or not the relationship is a linear function.**

The differences in x, Δx, are all 1, so they are constant.
The differences in y, Δy, are all 2.5, so they are constant.
Since the finite differences are all constant, the relationship is a linear function.

STEP 3 **Determine the slope, or rate of change, of the linear function.**

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{2.5}{1} = 2.5$$

STEP 4 **Determine the y-intercept of the linear function.**
Work backwards from $x = 1$ and $y = 7.5$.

x	y
0	b
1	7.5
2	10

$1 - 0 = 1$
$2 - 1 = 1$

$7.5 - b = 2.5$
$10 - 7.5 = 2.5$

$$\begin{aligned} 7.5 - b &= 2.5 \\ 7.5 - 7.5 - b &= 2.5 - 7.5 \\ -b &= -5 \\ b &= 5 \end{aligned}$$

The y-intercept is (0, 5).

STEP 5 **Use the slope and the y-coordinate of the y-intercept to write the function in slope-intercept form.**

$$y = mx + b$$
$$y = 2.5x + 5$$

YOU TRY IT! #3

For the data set below, determine if the relationship is a linear function. If so, determine a function, in slope-intercept form, relating the variables.

x	y
1	9
2	4
3	-1
4	-6
5	-11

EXAMPLE 4

For the data set below, determine if the relationship is a linear function. If so, write a function, in slope-intercept form, relating the variables.

x	y
2	22
4	21
6	20
8	19
10	18

STEP 1 **Determine the finite differences between successive x-values and successive y-values.**

	x	y	
$\Delta x = 4 - 2 = 2$	2	22	$\Delta y = 21 - 22 = -1$
$\Delta x = 6 - 4 = 2$	4	21	$\Delta y = 20 - 21 = -1$
$\Delta x = 8 - 6 = 2$	6	20	$\Delta y = 19 - 20 = -1$
$\Delta x = 10 - 8 = 2$	8	19	$\Delta y = 18 - 19 = -1$
	10	18	

STEP 2 **Determine whether or not the relationship is a linear function.**

The differences in x, Δx, are all 2, so they are constant.
The differences in y, Δy, are all −1, so they are constant.
Since the finite differences are all constant, the relationship is a linear function.

STEP 3 **Determine the slope of the linear function.**

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{-1}{2} = -\frac{1}{2}$$

STEP 4 **Determine the y-intercept of the linear function.**
Work backwards from $x = 2$ and $y = 22$.

x	y
0	b
2	22
4	21

$2 - 0 = 2$
$4 - 2 = 2$

$22 - b = -1$
$21 - 22 = -1$

$$\begin{aligned} 22 - b &= -1 \\ 22 - 22 - b &= -1 - 22 \\ -b &= -23 \\ b &= 23 \end{aligned}$$

The y-intercept is (0, 23).

STEP 5 **Use the slope and the y-coordinate of the y-intercept to write the function in slope-intercept form.**

$$y = mx + b$$
$$y = -\tfrac{1}{2}x + 23$$

YOU TRY IT! #4

For the data set below, determine if the relationship is a linear function. If so, determine a function relating the variables.

x	y
6	11
9	16
12	21
15	26
18	31

PRACTICE/HOMEWORK

For questions 1 - 4 determine the equation of the linear function with the given characteristics.

1. slope = 0.4, y-intercept = (0, −3)

2. slope = $\frac{2}{3}$, y-intercept = $(0, 3\frac{1}{3})$

3. slope = $-\frac{2}{5}$, contains the point (10, 3)

4. slope = $\frac{1}{4}$, contains the point (−8, 1)

For questions 5 - 16, determine whether or not the relationship shows a linear function. If the data set represents a linear function, write the equation for the function.

5.

x	y
1	1
2	4
3	9
4	16
5	25

6.

x	y
1	5.5
2	7.5
3	9.5
4	11.5
5	13.5

7.

x	y
1	8
2	11
3	14
4	17
5	20

8.

x	y
1	24
2	20
3	16
4	12
5	8

9.

x	y
0	1.7
1	1.1
2	0.5
3	-0.1
4	-0.7

10.

x	y
0	2
2	4
4	8
6	16
8	32

11.

x	y
2	4
4	5
6	7
8	10
10	14

12.

x	y
2	8
4	9
6	10
8	11
10	12

13.

x	y
3	2
5	10
7	18
9	26
11	34

14.

x	y
1	10
2	8
3	6
4	4
5	2

15.

x	y
1	16
2	15
3	13
4	10
5	6

16.

x	y
1	120
2	60
3	40
4	30
5	24

For questions 17 - 20, use the information in the problem to create a table of data. Then, use the table to determine if the situation is linear or not. If the situation is linear, then use the table to determine a linear function.

SCIENCE

17. The elevation of Lake Sam Rayburn is 164 feet above mean sea level. During the summer, if it does not rain, the elevation of the lake decreases by 0.5 feet each week.

x	y

18. A swimming pool has a capacity of 10,000 gallons of water. The swimming pool was about 20% full when a water hose was turned on to fill the pool at a rate of 75 gallons every 5 minutes.

x	y

19. According to a recent county health department survey, there were 750 mosquitos per acre in a county park. After a recent rainstorm, the number of mosquitos doubled every 2 days.

x	y

FINANCE

20. Marla has $85 in her savings. She earns $6.50 per hour after taxes and payroll deductions and plans to save half of what she earns each hour.

x	y

Modeling with Linear Functions

1.3

FOCUSING QUESTION How can you use finite differences to construct a linear model for a data set?

LEARNING OUTCOMES

- I can use finite differences to write a linear function that describes a data set.
- I can apply mathematics to problems that I see in everyday life, in society, and in the workplace.

ENGAGE

Mariette, a dendrochronologist, observed that some tree stumps have rings that are close together while other tree stumps have rings that are farther apart. Why does a tree stump have rings? What might cause the rings to be closer together or farther apart?

Image credit: Adrian Pingstone, Tree ring, Wikimedia Commons

EXPLORE

Each year during the growing season, trees grow larger by adding another layer of cells just beneath the bark. This layer is called a tree ring. Because a tree ring is added each year, scientists can determine the age of a tree by counting the number of tree rings that are present.

However, not all tree rings have the same width. Trees grow more when there is plenty of rain and the soil is fertile. Scientists can draw conclusions about temperature and rainfall for a particular year based on the width of the tree ring for that year.

Mariette measured the width of tree rings from a core sample she took from a post oak tree in the Brazos River valley of central Texas. From the tree ring width, she calculated the radius of the tree. The table below shows her results.

YEAR	2000	2001	2002	2003	2004	2005	2006
YEAR NUMBER	0	1	2	3	4	5	6
RADIUS (CM)	2.5	3.1	3.5	3.9	4.4	4.9	5.5

1. Calculate the finite differences between the year number and the radius.

2. Are the first differences in the radius constant? Explain how you know.

3. What is the average finite difference in radius?

4. Use the information from the table to write a function rule that models the data.

5. What do the slope and y-intercept from your function rule mean in the context of this situation?

6. Use your model to predict the radius of the tree in 2015.

7. In what year will the radius of the tree be 12.5 centimeters?

8. What would the circumference of the tree be in 2015?

9. Make a scatterplot of your data set and graph the function model over the scatterplot. How well would you say the function model predicts the actual values in the data set? Explain your reasoning.

REFLECT

- **How can you determine a linear function model for a data set if the first differences are not exactly the same, but are almost constant?**

- **Once you have your linear function model, how can you use the model to determine a value of the independent variable that generates a particular value of the dependent variable?**

EXPLAIN

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A linear function model can be used to represent sets of mathematical and real-world data. Dendrochronologists use core samples, or cylinders that are about 5 millimeters in diameter that are drilled into and extracted from the tree to measure the width of tree rings. Once they have their model, they can use different measures, such as circumference of a tree, to calculate the age of the tree.

You can use Mariette's data to generate a model relating the circumference of a tree to the age of the tree. For trees that were planted in 2000, the table shows the growth rate.

YEAR	2000	2001	2002	2003	2004	2005	2006
YEAR NUMBER	0	1	2	3	4	5	6
RADIUS (CM)	2.5	3.1	3.5	3.9	4.4	4.9	5.5

Add a new row to the table to calculate the circumference. Recall that circumference can be calculated using the formula $C = 2\pi r$, where r represents the radius of the circle and C represents the circumference of the circle. Round the circumference to the nearest tenth if necessary.

YEAR	2000	2001	2002	2003	2004	2005	2006
YEAR NUMBER	0	1	2	3	4	5	6
RADIUS (CM)	2.5	3.1	3.5	3.9	4.4	4.9	5.5
CIRCUMFERENCE (CM)	15.7	19.5	22.0	24.5	27.6	30.8	34.5

Use the rows for year number and circumference to calculate the first finite differences.

+1 +1 +1 +1 +1 +1

YEAR NUMBER	0	1	2	3	4	5	6
CIRCUMFERENCE (CM)	15.7	19.5	22.0	24.5	27.6	30.8	34.5

+3.8 +2.5 +2.5 +3.1 +3.2 +3.7

These first finite differences are not equal, but are all close to +3. Calculate the average finite difference, and use that to determine the slope of the linear function model.

$$\Delta y = \frac{3.8 + 2.5 + 2.5 + 3.1 + 3.2 + 3.7}{6} \approx 3.13$$

$$\Delta x = 1$$

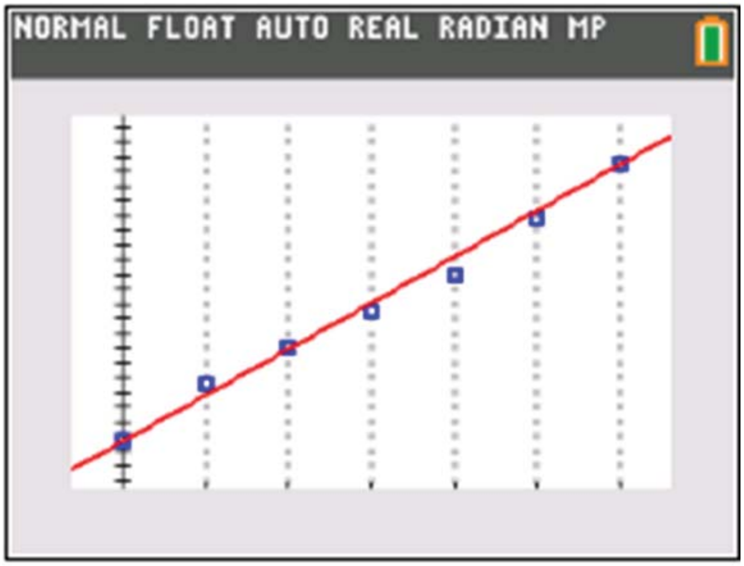
$$\frac{\Delta y}{\Delta x} = \frac{3.13}{1} = 3.13$$

Using the slope and y-intercept, you can write the function model, $f(x) = 15.7 + 3.13x$. Once you have a function model, you can use that model to make predictions.

MODELING WITH LINEAR FUNCTIONS

Real-world data rarely follows exact patterns, but you can use patterns in data to look for trends. If the data increases or decreases at about the same rate, then a linear function model may be appropriate for the data set.

You can also use a scatterplot and a graph to show how the values in the data set are related to the function model. The graph of the function model could also be useful in making predictions from the model.



$f(x) = 15.7 + 3.13x$

EXAMPLE 1

A student takes small steps away from a motion detector at an approximately constant rate. The time, in seconds, for which the student walks and the distance, in meters, the student walks are recorded in the table.

TIME (S)	0	1	2	3	4
DISTANCE (M)	0.25	0.85	1.55	2.2	2.75

Generate a linear function model for this situation. Based on your model, how far away will the student be from the motion detector after 10 seconds?

STEP 1 Calculate the finite differences in the table.

+1 +1 +1 +1

TIME (S)	0	1	2	3	4
DISTANCE (M)	0.25	0.85	1.55	2.2	2.75

+0.6 +0.7 +0.65 +0.55

STEP 2 Calculate the average finite difference in the table and use this to determine the slope, or average velocity, for a linear function model.

$$\frac{0.6 + 0.7 + 0.65 + 0.55}{4} = \frac{2.5}{4} = 0.625$$

STEP 3 Use the slope and y-intercept to write a linear function model.

$$y = 0.625x + 0.25$$

STEP 4 Use your linear function model to make a prediction.

$$y = 0.625(10) + 0.25$$
$$y = 6.25 + 0.25$$
$$y = 6.5$$

According to the linear function model, the student will be 6.5 meters away from the motion detector after 10 seconds.

YOU TRY IT! #1

Tracy, a long distance runner, times herself as she runs a half-marathon, which is 13.1 miles long. The distance is measured in miles and the time is measured in minutes.

DISTANCE (MI)	1	2	3	4	5
TIME (MIN)	8.5	16.2	24.6	33.1	41.8

Generate a linear function model for this situation. Based on your model, how long to the nearest minute will it take Tracy to run the half-marathon?

EXAMPLE 2

Engineers conducting experiments in accident reconstruction want to see how long it would take a vehicle on a highway to coast to a stop if the brakes were inoperable. The first few seconds of the experiment are recorded in the table below. Time is measured in seconds and speed is measured in miles per hour.

TIME (S)	1	2	3	4	5
SPEED (MPH)	65	62	58	56	53

Generate a linear function model for this situation. Based on your model, when will the car come to a stop, to the nearest second?

STEP 1 **Calculate the first finite differences in the table.**

+1 +1 +1 +1

TIME (S)	1	2	3	4	5
SPEED (MPH)	65	62	58	56	53

-3 -4 -2 -3

STEP 2 Calculate the average first finite difference in the table and use this to determine the slope, or average acceleration, for a linear function model.

$$\frac{-3+(-4)+(-2)+(-3)}{4} = \frac{-12}{4} = -3$$

STEP 3 Use the average acceleration and first finite differences in the x-values to determine the y-intercept of a linear function model.

+1

TIME	0	1
SPEED	b	65

$65 - b = -3$

$$\begin{aligned} 65 - b &= -3 \\ 65 - b + 3 &= -3 + 3 \\ 68 - b &= 0 \\ 68 - b + b &= 0 + b \\ 68 &= b \end{aligned}$$

STEP 4 Write a linear function model.

$$y = -3x + 68$$

STEP 5 Use the linear function model to make a prediction.

$$\begin{aligned} 0 &= -3x + 68 \\ 0 - 68 &= -3x + 68 - 68 \\ -68 &= -3x \\ (-68) \div (-3) &= (-3x) \div (-3) \\ 22.667 &\approx x \end{aligned}$$

According to the linear function model, the car will come to a stop after approximately 23 seconds.

YOU TRY IT! #2

Caleb's parents set up a checking account for him before college so that he will be able to pay the utilities for his apartment. Caleb keeps track of his spending in the table below. Time represents the number of months he has been in his apartment and the checking account balance is measured in dollars.

TIME (MONTHS)	0	1	2	3	4
BALANCE	$1550	$1355	$1170	$978	$791

Generate a linear function model for this situation. Based on your model, when will the checking account balance dip below $100?

PRACTICE/HOMEWORK

For the following sets of data, calculate the average finite difference, and use that to determine the slope of a linear function that could model the data.

1.

x	y
1	15.3
2	25.3
3	35.2
4	45.4
5	55.2
6	65.3

2.

x	0	1	2	3	4	5	6
y	50	47.2	44.3	41.5	38.5	35.6	32.5

3.

x	1	2	3	4	5
y	14.25	14.05	15.35	16	16.55

For problems 4 – 6, determine a linear function to model the situation.

FINANCE

4. Madeleine has a gift card to her favorite coffee shop. The table below shows how much is remaining on the gift card after each purchase at the coffee shop.

PURCHASES	0	1	2	3	4	5
BALANCE (DOLLARS)	40	35.68	31.22	26.97	22.65	18.40

SCIENCE

5. Gus records the mileage on his car, so he can determine his average mileage per month. Below are some of his collected data.

TIME (MONTHS)	1	2	3	4	5	6
MILEAGE (MILES)	11,540	12,482	13,570	14,670	15,682	16,757

FINANCE

6. David is purchasing apps for his cell phone. The table below shows how his total cost changes with each app that he selects.

NUMBER OF APPS PURCHASED	1	2	3	4	5
COST (DOLLARS)	1.25	2.50	3.75	5.00	6.25

Use the following situation to answer problems 7 – 10.

SCIENCE

Charlie is measuring his little brother's height throughout the year to see how much he grows. The table below shows how his height changes during the first 5 months.

TIME (MONTHS)	0	1	2	3	4	5
HEIGHT (INCHES)	54	54.20	54.45	54.85	55.10	55.25

7. Write a function rule to model the situation.

8. What do the slope and y-intercept from your function rule mean in the context of this situation?

9. Use your model to predict the height of Charlie's brother after a year.

10. In what month will his height be approximately 56 inches?

Use the following situation to answer problems 11 – 13.

CRITICAL THINKING

Jeff noticed that the nutrition information on his box of cereal states that there are 14 servings in the cereal box. He decided to put their claim to the test. He recorded the weight of the remaining cereal after each serving, as shown in the table below.

NUMBER OF SERVINGS	0	1	2	3	4	5
WEIGHT OF REMAINING CEREAL (OUNCES)	14	12.7	11.6	10.7	9.5	8.4

11. Write a function rule that models the situation.

12. What do the slope and y-intercept from your function rule mean in the context of this situation?

13. Was Jeff able to confirm the claim on the cereal box by eating 14 servings? Explain your answer.

Use the following situation to answer problems 14 – 17.

SCIENCE

Bob is tracking a hurricane moving toward the coast of Florida. The table below shows its distance from land over time.

TIME (HOURS)	0	1	2	3	4
DISTANCE (MILES)	704	684	663.7	644.2	624.4

14. Write a function rule that models the situation.

15. What do the slope and y-intercept from your function rule mean in the context of this situation?

16. About how far will the hurricane be from land after 24 hours?

17. Approximately when will the hurricane make landfall?

Use the following situation to answer problems 18 – 19.

CRITICAL THINKING

Maddie is running a 10-K (10 kilometer) race. She wears an electronic chip that tracks her progress throughout the race. She runs at a fairly steady pace throughout the race, as shown in her chip data below.

DISTANCE (KILOMETERS)	0	1	2	3	4	5
TIME (MINUTES)	0	4.1	7.9	11.8	15.5	19

18. Write a function rule that models the situation.

19. If Maddie continues at this rate, will she beat her previous best time of 37.5 minutes? Explain your answer.

Use the following situation to answer problems 20 – 21.

FINANCE

Nikki has a job as a waitress where she gets an hourly wage plus tips. The table below shows her total earnings for working one weekend.

TIME WORKED (HOURS)	1	2	3	4	5
TOTAL EARNED (DOLLARS)	9.3	19.3	30.73	43.23	56.33

Nikki calculates that the function $f(x) = 11.8x - 2.5$ models her earnings over time. She understands that the slope of 11.8 means she earned an average of about \$11.80 per hour. However, she is uncertain about why she has a negative y-intercept in her function equation, since she didn't earn -\$2.50 for working 0 hours.

20. Is her equation correct? Explain why or why not.

21. Since she earned \$0 for working zero hours, she now decides to include the point (0, 0) in her data set. How will this affect her function equation to model the situation?

Writing Exponential Functions

1.4

FOCUSING QUESTION What are the characteristics of an exponential function?

LEARNING OUTCOMES

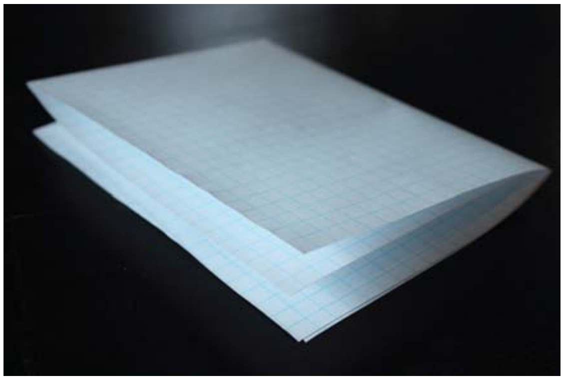
- I can determine patterns that identify an exponential function from its related common ratios.
- I can classify a function as linear or exponential when I am given a table.
- I can determine the exponential function from a table using common ratios, including any restrictions on the domain and range.
- I can analyze patterns to connect the table to a function rule and communicate the exponential pattern as a function rule.

ENGAGE

Miranda shared a cookie recipe on social media with three friends. Each of Miranda's friends shared the cookie recipe with three of their friends. If this trend continues, how many people will receive a cookie recipe in the fifth round?

EXPLORE

Begin with a sheet of paper. Fold it in half and record the number of layers of paper after the fold in a table like the one shown.



NUMBER OF FOLDS	NUMBER OF LAYERS
0	1
1	
2	
3	
4	
5	
6	

1. What is the difference between the numbers of folds in consecutive rows in the table?

2. What is the difference between the numbers of layers in consecutive rows in the table?

NUMBER OF FOLDS	NUMBER OF LAYERS
0	1
1	
2	
3	
4	
5	
6	

3. Are the finite differences between the number of layers and the number of folds constant? How can you tell?

4. What patterns do you observe in the differences in the table?

5. Is this a linear relationship? How do you know?

6. What is the ratio between successive numbers of layers?

7. How many layers would there be after the 7th fold? 10th fold?

8. What type of function best describes this relationship? Explain your reasoning.

9. Write an equation that could be used to determine y, the number of layers, if you know x, the number of folds.

Begin with a new sheet of paper.

10. What is area of the sheet of paper without any folds?

11. Fold the paper in half and record the area of the region showing after the fold in a table like the one shown. If necessary, round to the nearest tenth of a square inch.

NUMBER OF FOLDS	AREA OF REGION
0	
1	
2	
3	
4	
5	
6	

12. What is the difference between the numbers of folds in consecutive rows in the table?

13. What is the difference between the areas of regions in consecutive rows in the table?

NUMBER OF FOLDS	AREA OF REGION
0	
1	
2	
3	
4	
5	
6	

14. Are the finite differences between the area of the regions and the number of folds constant? How can you tell?

15. What patterns do you observe in the differences in the table?

16. Is this a linear relationship? How do you know?

17. What is the ratio between successive areas of regions?

18. What would be the area of the region present after the 7th fold?

19. What type of function best describes this relationship? Explain your reasoning.

20. Write an equation that could be used to determine y, the area of the region, if you know x, the number of folds.

REFLECT

- **What do you notice about the successive ratios in each relationship?**
- **What relationship exists between the successive ratios in the dependent variable and the equations that you have written?**

EXPLAIN

When you found finite differences that were constant for the dependent variable, y, and the differences between values of the independent variable, x, were the same, the relationship was linear. But as you have seen, this is not true for every functional relationship.

If the finite differences are not constant, look at the ratios between successive rows in the table. If these ratios are constant, then the constant ratio is called a **common ratio** and the relationship is an **exponential function**. An exponential relationship is

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one in which there is repeated multiplication. For example, a geometric sequence has a constant multiplier, so the sequence can be represented as an exponential function with a domain of whole numbers.

Let's look more closely at an exponential function. The table below shows the relationship between x and $f(x)$. In an exponential function, $f(x) = ab^x$, b represents the base of the exponential function, which is also a common ratio or constant multiplier. The parameter a represents an initial value or y-intercept.

In an exponential function, if the base is greater than 1, then the function represents **exponential growth**. If the base is between 0 and 1, then the function represents **exponential decay**.

x	$y = f(x)$
0	a
1	ab
2	$ab(b)$
3	$ab(b)(b)$
4	$ab(b)(b)(b)$
5	$ab(b)(b)(b)(b)$

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta y = ab - a = a(b - 1)$

$\Delta y = ab(b) - ab = ab(b - 1)$

$\Delta y = ab(b)(b) - ab(b) = ab^2(b - 1)$

$\Delta y = ab(b)(b)(b) - ab(b)(b) = ab^3(b - 1)$

$\Delta y = ab(b)(b)(b)(b) - ab(b)(b)(b) = ab^4(b - 1)$

Unlike a linear function, the finite differences for an exponential function, $f(x) = ab^x$, are not constant. Instead, the multiplicative pattern that is present in the original data repeats in the finite differences.

Instead of looking at the finite differences, for an exponential function, take a closer look at the successive ratios.

x	$y = f(x)$
0	a
1	ab
2	$ab(b)$
3	$ab(b)(b)$
4	$ab(b)(b)(b)$
5	$ab(b)(b)(b)(b)$

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$$\frac{y_n}{y_{n-1}} = \frac{ab}{a} = b$$

$$\frac{y_n}{y_{n-1}} = \frac{a(b)(b)}{a(b)} = b$$

$$\frac{y_n}{y_{n-1}} = \frac{a(b)(b)(b)}{a(b)(b)} = b$$

$$\frac{y_n}{y_{n-1}} = \frac{a(b)(b)(b)(b)}{a(b)(b)(b)} = b$$

$$\frac{y_n}{y_{n-1}} = \frac{a(b)(b)(b)(b)(b)}{a(b)(b)(b)(b)} = b$$

Notice that in the table, $\Delta x = 1$. Knowing that, the common ratio for successive rows is equivalent to b, which is the base of the exponential relationship.

COMMON RATIOS AND EXPONENTIAL FUNCTIONS

In an exponential function, the ratios between successive y-values, $\frac{y_n}{y_{n-1}}$, are constant if the differences between successive x-values, Δx, are also constant.

If the ratios of successive values of the dependent variable in a table of values are constant, then the values represent an exponential function.

You can also use the common ratio to write an exponential function describing the relationship between the independent and dependent variables.

EXAMPLE 1

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	1
1	5
2	25
3	125
4	625

STEP 1 **Determine the finite differences between successive x-values and the ratios between successive y-values.**

x	y
0	1
1	5
2	25
3	125
4	625

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\frac{y_n}{y_{n\text{-}1}} = \frac{5}{1} = 5$

$\frac{y_n}{y_{n\text{-}1}} = \frac{25}{5} = 5$

$\frac{y_n}{y_{n\text{-}1}} = \frac{125}{25} = 5$

$\frac{y_n}{y_{n\text{-}1}} = \frac{625}{125} = 5$

STEP 2 **Determine whether or not the differences between successive x-values and ratios between successive y-values are constant.**

The differences between successive values of x, Δx, are all 1, so they are constant.

The ratios between successive values of y, $\frac{y_n}{y_{n\text{-}1}}$, are all 5, so they are constant.

$\frac{\frac{y_n}{y_{n\text{-}1}}}{\Delta x} = \frac{5}{1} = 5$ for all pairs of Δx and $\frac{y_n}{y_{n\text{-}1}}$.

STEP 3 **Determine whether or not the set of data represents an exponential function.**

Yes, the set of data represents an exponential function because the differences between successive x-values and the ratios between successive y-values are constant.

YOU TRY IT! #1

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	1.2
1	1.44
2	1.728
3	2.0736

EXAMPLE 2

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	8
1	4
2	2
3	0.8
4	0.4

STEP 1 Determine the finite differences between successive x-values and ratios between successive y-values.

x	y
0	8
1	4
2	2
3	0.8
4	0.4

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\frac{y_n}{y_{n\text{-}1}} = \frac{4}{8} = \frac{1}{2}$

$\frac{y_n}{y_{n\text{-}1}} = \frac{2}{4} = \frac{1}{2}$

$\frac{y_n}{y_{n\text{-}1}} = \frac{0.8}{2} = \frac{2}{5}$

$\frac{y_n}{y_{n\text{-}1}} = \frac{0.4}{0.8} = \frac{1}{2}$

STEP 2 Determine whether or not the differences are constant.

The differences in x, Δx, are all 1, so they are constant.

The ratios of successive values of y, $\frac{y_n}{y_{n\text{-}1}}$, are not all the same, so they are not constant.

STEP 3 Determine whether or not the set of data represents an exponential function.

No, the set of data does not represent an exponential function because even though the differences in successive values of x are constant, the ratios between successive values of y are not constant.

YOU TRY IT! #2

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	9.2
1	18.4
2	46
3	92
4	230

EXAMPLE 3

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	4
2	10
3	25
4	62.5
5	156.25

STEP 1 **Determine the finite differences between successive x-values and the ratios between successive y-values.**

x	y
1	4
2	10
3	25
4	62.5
5	156.25

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\frac{y_n}{y_{n\text{-}1}} = \frac{10}{4} = 2.5$

$\frac{y_n}{y_{n\text{-}1}} = \frac{25}{10} = 2.5$

$\frac{y_n}{y_{n\text{-}1}} = \frac{62.5}{25} = 2.5$

$\frac{y_n}{y_{n\text{-}1}} = \frac{156.25}{62.5} = 2.5$

STEP 2 **Determine whether or not the relationship is an exponential function.**

The differences in x, Δx, are all 1, so they are constant.

The ratios between successive values of $y, \frac{y_n}{y_{n\text{-}1}}$, are all 2.5, so they are constant.

Since the first differences in x and the successive ratios in y are all constant, the relationship is an exponential function.

STEP 3 **Determine the y-intercept of the exponential function.**

Work backwards from $x = 1$ and $y = 4$.

x	y
0	a
1	4
2	10

$1 - 0 = 1$

$2 - 1 = 1$

$\frac{4}{a} = 2.5$

$\frac{10}{4} = 2.5$

$$\frac{4}{a} = 2.5$$
$$a\left(\frac{4}{a}\right) = 2.5(a)$$
$$4 = 2.5a$$
$$\frac{4}{2.5} = \frac{2.5a}{2.5}$$
$$1.6 = a$$

The y-intercept is (0, 1.6).

STEP 4 **Use the y-coordinate of the y-intercept and the common ratio to write the exponential function.**

$y = 1.6(2.5)^x$

YOU TRY IT! #3

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	14
2	98
3	686
4	4802

EXAMPLE 4

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	1000
2	400
3	160
4	64
5	25.6

STEP 1 **Determine the first differences between successive x-values and the ratios between successive y-values.**

x	y
1	1000
2	400
3	160
4	64
5	25.6

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{400}{1000} = 0.4$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{160}{400} = 0.4$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{64}{160} = 0.4$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{25.6}{64} = 0.4$$

STEP 2 **Determine whether or not the relationship is an exponential function.**

The differences in x, Δx, are all 1, so they are constant.

The ratios between successive values of y, $\frac{y_n}{y_{n\text{-}1}}$, are all 0.4, so they are constant.

Since the first differences in x and the successive ratios in y are all constant, the relationship is an exponential function.

STEP 3 **Determine the y-intercept of the exponential function.**

Work backwards from $x = 1$ and $y = 1000$.

x	y
0	a
1	1000
2	400

$1 - 0 = 1$

$2 - 1 = 1$

$\frac{1000}{a} = 0.4$

$\frac{400}{1000} = 0.4$

$$
\begin{aligned}
\frac{1000}{a} &= 0.4 \\
a\left(\frac{1000}{a}\right) &= 0.4(a) \\
1000 &= 0.4a \\
\frac{1000}{0.4} &= \frac{0.4a}{0.4} \\
2500 &= a
\end{aligned}
$$

The y-intercept is (0, 2500).

STEP 4 **Use the y-coordinate of the y-intercept and the common ratio to write the exponential function.**

$y = 2500(0.4)^x$

YOU TRY IT! #4

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	81
2	27
3	9
4	3
5	1

PRACTICE/HOMEWORK

For questions 1-4 use finite differences to determine if each table represents an exponential function. Use real objects such as pennies or beans to create 4 stacks, one to represent the y-value for each x-value, and use the stacks to help you determine whether or not the data represents an exponential function.

1.

x	y
0	2
1	6
2	18
3	54

2.

x	y
0	3
1	4
2	7
3	12

3.

x	y
0	0
1	1
2	8
3	27

4.

x	y
0	3
1	6
2	12
3	24

For questions 5-8 identify if each table represents an exponential function or not. If the table represents an exponential function, identify the common ratio.

5.

x	y
1	2
2	4
3	6
4	8

Exponential Function?
Common Ratio:

6.

x	y
1	2
2	4
3	8
4	16

Exponential Function?
Common Ratio:

7.

x	y
1	3
2	4.5
3	6.75
4	10.125

Exponential Function?
Common Ratio:

8.

x	y
1	4
2	1
3	0.25
4	0.0625

Exponential Function?
Common Ratio:

For questions 9-12 use the situation below.

CRITICAL THINKING

A sheet of paper is 0.1 mm thick. When the paper is folded in half, the total thickness of the layers of paper is 0.2 mm. When the paper is folded in half again, the total thickness of the layers of paper is 0.4 mm.

9. Complete the table below to represent the situation.

NUMBER OF FOLDS x	TOTAL THICKNESS OF LAYERS y
0	0.1
1	0.2
2	
3	
4	

10. Does the situation represent a linear function or an exponential function? Justify your answer.

11. Which of the following represents the function that models this situation?

A. $y = x + 0.1$

B. $y = 2 \bullet 0.1^x$

C. $y = 0.1 \bullet 2^x$

D. $y = 2^x + 0.1$

12. Which of the following statements are true about the situation?

- $\Delta x = 1$
- The situation is an example of exponential decay.
- The function is increasing.
- The common ratio is 2.
- The y-intercept is (0, 0.1).
- The function is linear.
- The function is decreasing.
- $\Delta y = 0.1$
- The common ratio is 0.2.
- The situation is an example of exponential growth.

*For questions 13-18 identify if each table represents an exponential function or not. If the table **represents** an exponential function, write the function relating the variables.*

13.

x	y
0	0
1	4
2	32
3	108

Exponential Function?
Function:

14.

x	y
0	40
1	8
2	1.6
3	0.32

Exponential Function?
Function:

15.

x	y
0	50
1	25
2	12.5
3	6.25

Exponential Function?
Function:

16.

x	y
1	300
2	150
3	100
4	75

Exponential Function?
Function:

17.

x	y
1	4500
2	6750
3	10,125
4	15,187.5

Exponential Function?
Function:

18.

x	y
1	14
2	56
3	224
4	896

Exponential Function?
Function:

For questions 19-20 use the situation below.

CRITICAL THINKING

A sheet of paper has an area of 100 square inches. When the paper is cut in half, the area of one piece is 50 square inches. When that piece is cut in half, the area of one piece is 25 square inches.

NUMBER OF CUTS x	AREA OF ONE PIECE y
0	100
1	50
2	25

19. What would be the area of one piece after 5 cuts?

20. Write the function relating the variables.

21. Draw a diagram of the paper and how it is cut in half. Use the diagram to interpret the values of a and b in your exponential function. Communicate your mathematical reasoning and its implications using the diagram.

Modeling with Exponential Functions

1.5

FOCUSING QUESTION How can you use common ratios to construct an exponential model for a data set?

LEARNING OUTCOMES

- I can use finite differences or common ratios to classify a function as either linear or exponential when I am given a table of values.
- I can use common ratios to write an exponential function that describes a data set.
- I can apply mathematics to problems that I see in everyday life, in society, and in the workplace.

ENGAGE

DeAnna dropped a basketball and let it bounce several times. What would a graph of the height of the basketball versus time look like?

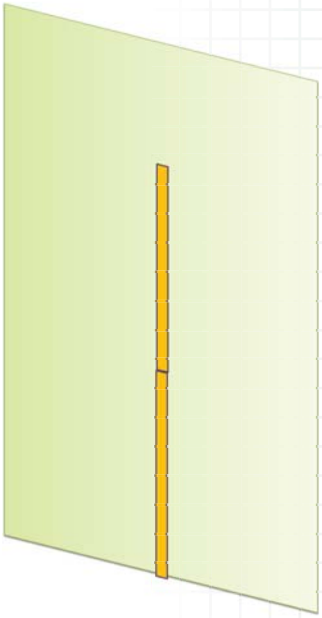
EXPLORE

According to international basketball guidelines, a basketball should be inflated such that when dropped from a height of 1.8 meters, the ball should bounce and rebound to a height of at least 1.2 meters but no more than 1.4 meters.

Bounce height is an important aspect to making sure that a ball used in any sport is properly inflated or is not worn out.

DIRECTIONS

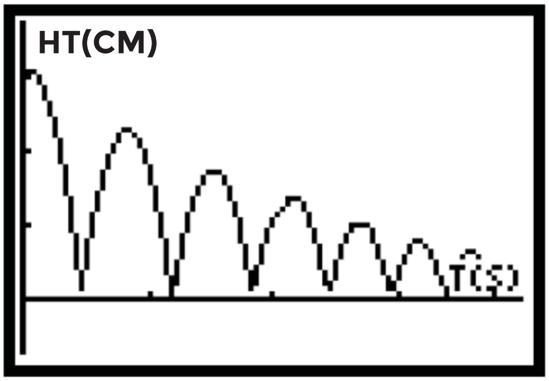
- Tape two meter sticks to the wall and use them to measure the bounce height, or the height to which the ball bounces, when dropped from a given height.
- Begin with 180 centimeters. Select a spot on the ball, such as the highest point on the top of the ball, to use as a consistent reference point. Drop the ball from this height and record the height of the first bounce.
- Repeat for two more trials and calculate the average bounce height for a drop from 180 centimeters.



- Use the average bounce height as the starting point for the second bounce. Record the bounce height when the ball is released from this drop height for three trials. Calculate the average bounce height.
- Repeat for a total of 6 bounces.
- Record your information in a table like the one shown.

DROP HEIGHT (CM)	BOUNCE HEIGHT 1 (CM)	BOUNCE HEIGHT 2 (CM)	BOUNCE HEIGHT 3 (CM)	AVERAGE BOUNCE HEIGHT (CM)
180				

1. If you were to drop the ball once and let it continue bouncing until it stopped, a height versus time graph of the ball might look like the figure shown. Use an ordered pair (bounce number, height of bounce) to label each point shown on the graph.



2. Calculate the finite differences between the bounce number and the average bounce height. Do the data appear to be linear? How do you know?

3. Calculate the ratios between the average bounce heights for successive bounces. Do the data appear to be exponential? How do you know?

4. What is the starting point, or *y*-intercept, of the data?

5. Use either the finite differences or common ratios to determine a function that best models the relationship between the bounce number, x, and the average bounce height, $f(x)$. If it is a linear function, use slope-intercept form. If it is an exponential function, use the form $f(x) = ab^x$.

6. What do the y-intercept and either rate of change or base from your function rule mean in the context of this situation?

7. Use your model to predict the height of the 8th bounce.

8. What is the diameter of the ball that you used? (You may need to use the formula $C = \pi d$ to calculate the diameter of the ball.)

9. Thinking about the diameter of the ball, what will be the last bounce observed before the ball bounces to a height that is less than the diameter?

REFLECT

- **How can you determine an exponential function model for a data set if the common ratios are not exactly the same, but are very close to each other?**
- **Once you have determined your exponential function model, how can you use the model to determine a value of the independent variable that generates a particular value of the dependent variable?**

EXPLAIN

Watch Explain and You Try It Videos

or click here

Exponential function models can be used to represent sets of mathematical and real-world data. If an exponential function is decreasing, then the function is called an **exponential decay** function, since the values of the dependent variable, $f(x)$, decay, or become smaller, as the values of the independent variable, x, increase. The relationship between bounce height and drop height is an exponential decay relationship.

Other exponential functions are increasing and are called **exponential growth** functions. The values of the dependent variable, $f(x)$, grow, or become larger, as the values of the independent variable, x, increase. Population growth is sometimes exponential when the population of a city or county grows at the same percent each year.

For example, the table below shows the population of Hays County, Texas, for certain years since 1985.

Instead of looking at the finite differences, for an exponential function, take a closer look at the successive ratios.

5-YEAR INTERVAL, x	YEAR	POPULATION, $f(x)$
0	1985	56,225
1	1990	65,767
2	1995	78,956
3	2000	99,070
4	2005	126,470
5	2010	158,289
6	2015	197,298

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

$\frac{y_n}{y_{n\text{-}1}} = \frac{65{,}767}{56{,}225} \approx 1.17$

$\frac{y_n}{y_{n\text{-}1}} = \frac{78{,}956}{65{,}767} \approx 1.20$

$\frac{y_n}{y_{n\text{-}1}} = \frac{99{,}070}{78{,}956} \approx 1.25$

$\frac{y_n}{y_{n\text{-}1}} = \frac{126{,}470}{99{,}070} \approx 1.28$

$\frac{y_n}{y_{n\text{-}1}} = \frac{158{,}289}{126{,}470} \approx 1.25$

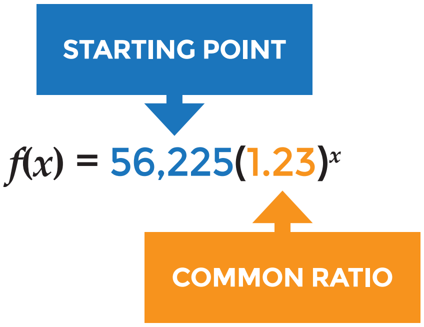
$\frac{y_n}{y_{n\text{-}1}} = \frac{197{,}298}{158{,}289} \approx 1.25$

Data Source: U.S. Census Bureau and Texas Department of State Health Services

The successive ratios are not equal, but are all close to the same value, 1.25. Calculate the average ratio and use that as the base, b, for the exponential function model.

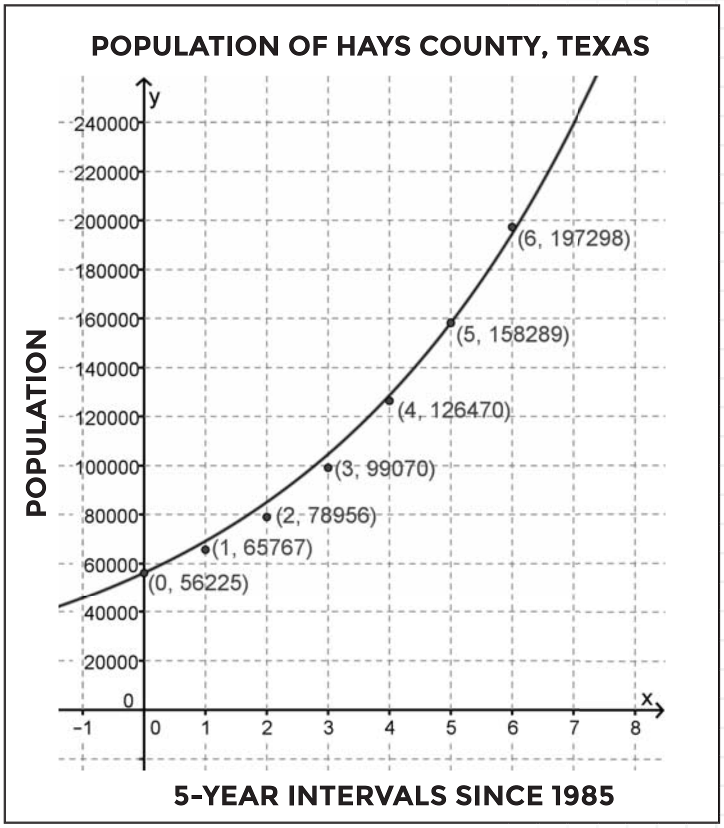
$$\frac{1.17 + 1.20 + 1.25 + 1.28 + 1.25 + 1.25}{6} \approx 1.23$$

Use the initial population for 5-Year Interval 0, which is 56,225, as the starting point, a, to write the function, $f(x) = 56{,}225(1.23)^x$. Once you have a function model, you can use that model to make predictions.



$$f(x) = 56{,}225(1.23)^x$$

Make a scatterplot of the data and graph the function rule over the scatterplot.



Notice that not all of the data points lie on the curve representing the function model. That is because the successive ratios are not exactly equal, but are close to 1.23.

Community leaders need to know how many people will live in the community in order to decide how many fire stations, schools, or restaurants to build. Demographers, or people who study population trends, use population models like this one to make predictions about how many people will live in a community in a particular year. Community and business leaders rely on demographers in order to help them make better decisions for people living in a particular community.

A graph helps you to visualize the data in order to make predictions. For example, interval 7 represents seven 5-year intervals, or 35 years, since 1985. 1985 + 35 = 2020. The population model contains the point (7, 240,000) which means that at interval 7, or the year 2020, the population of Hays County, Texas, could be 240,000.

MODELING WITH EXPONENTIAL FUNCTIONS

Real-world data rarely follows exact patterns, but you can use patterns in data to look for trends. If the data increases or decreases with approximately the same ratio, then an exponential function model may be appropriate for the data set.

Of course, not all exponential relationships involve growth. Automobile depreciation is a loss in the value of an automobile over time. If the value of an automobile loses a percent of its value each year, then the depreciation is an exponential decay.

EXAMPLE 1

Approximate radiation levels, in millirads per hour, near the Fukushima nuclear power plant near Naraha, Japan are shown in the table below.

4-DAY INTERVAL, x	DATE	RADIATION LEVEL, $f(x)$
0	MARCH 22, 2011	71.2
1	MARCH 26, 2011	39.8
2	MARCH 30, 2011	22.3
3	APRIL 3, 2011	12.5
4	APRIL 7, 2011	7.1
5	APRIL 11, 2011	3.9
6	APRIL 15, 2011	2.2

Data Source: U.S. Department of Energy

Use the data set to determine if the relationship is linear or exponential.

STEP 1 **Determine the finite differences in values of x and values of $f(x)$.**

4-DAY INTERVAL, x	DATE	RADIATION LEVEL, $f(x)$
0	MARCH 22, 2011	71.2
1	MARCH 26, 2011	39.8
2	MARCH 30, 2011	22.3
3	APRIL 3, 2011	12.5
4	APRIL 7, 2011	7.1
5	APRIL 11, 2011	3.9
6	APRIL 15, 2011	2.2

$\Delta x = 1 - 0 = 1$ $\quad \Delta f(x) = 39.8 - 71.2 = -31.4$

$\Delta x = 2 - 1 = 1$ $\quad \Delta f(x) = 22.3 - 39.8 = -17.5$

$\Delta x = 3 - 2 = 1$ $\quad \Delta f(x) = 12.5 - 22.3 = -9.8$

$\Delta x = 4 - 3 = 1$ $\quad \Delta f(x) = 7.1 - 12.5 = -5.4$

$\Delta x = 5 - 4 = 1$ $\quad \Delta f(x) = 3.9 - 7.1 = -3.2$

$\Delta x = 6 - 5 = 1$ $\quad \Delta f(x) = 2.2 - 3.9 = -1.7$

STEP 2 Determine the ratios between successive values of $f(x)$.

4-DAY INTERVAL, x	DATE	RADIATION LEVEL, $f(x)$
0	MARCH 22, 2011	71.2
1	MARCH 26, 2011	39.8
2	MARCH 30, 2011	22.3
3	APRIL 3, 2011	12.5
4	APRIL 7, 2011	7.1
5	APRIL 11, 2011	3.9
6	APRIL 15, 2011	2.2

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

$\frac{y_n}{y_{n-1}} = \frac{39.8}{71.2} \approx 0.559$

$\frac{y_n}{y_{n-1}} = \frac{22.3}{39.8} \approx 0.560$

$\frac{y_n}{y_{n-1}} = \frac{12.5}{22.3} \approx 0.561$

$\frac{y_n}{y_{n-1}} = \frac{7.1}{12.5} \approx 0.568$

$\frac{y_n}{y_{n-1}} = \frac{3.9}{7.1} \approx 0.549$

$\frac{y_n}{y_{n-1}} = \frac{2.2}{3.9} \approx 0.564$

STEP 3 Determine whether the finite differences or the ratios between successive values of $f(x)$ are approximately constant.

- The finite differences range in value from –31.4 to –1.7. This is a wide range, so the finite differences are not even approximately constant.
- The ratios between successive values of $f(x)$ range from 0.549 to 0.568. These values are all close together, so the ratios between successive values of $f(x)$ are approximately constant.

The set of data represents an exponential function, rather than a linear function, because the differences in x are constant and the ratios between successive values of f(x) are approximately constant.

YOU TRY IT! #1

A major league baseball player's average in successive seasons is recorded in the table.

1-YEAR INTERVAL, x	YEAR	BATTING AVERAGE, $f(x)$
0	2009	0.213
1	2010	0.242
2	2011	0.271
3	2012	0.301
4	2013	0.330
5	2014	0.359

Determine whether the relationship is linear or exponential.

EXAMPLE 2

The table below shows the average viewership for the Super Bowl, in numbers of households.

3-YEAR INTERVAL, x	YEAR	VIEWERSHIP, $f(x)$
0	1970	23,050
1	1973	27,670
2	1976	29,440
3	1979	35,090
4	1982	40,020

Data Source: http://www.nielsen.com/us/en.html

Generate an exponential function model for Super Bowl viewership. How many households does your model predict will watch the Super Bowl in the year 2018.

STEP 1 Determine the finite differences in x-values and the ratios between successive values of $f(x)$.

3-YEAR INTERVAL, x	YEAR	VIEWERSHIP, $f(x)$
0	1970	23,050
1	1973	27,670
2	1976	29,440
3	1979	35,090
4	1982	40,020

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\frac{y_n}{y_{n-1}} = \frac{27,670}{23,050} \approx 1.20$

$\frac{y_n}{y_{n-1}} = \frac{29,440}{27,670} \approx 1.06$

$\frac{y_n}{y_{n-1}} = \frac{35,090}{29,440} \approx 1.19$

$\frac{y_n}{y_{n-1}} = \frac{40,020}{35,090} \approx 1.14$

STEP 2 Calculate the average of the ratios and use this value for the base, b, in your exponential function model.

$$\frac{1.20 + 1.06 + 1.19 + 1.14}{4} \approx 1.15$$

STEP 3 Using a, the initial viewership value from the table, and the calculated value for b, write an exponential function model.

$f(x) = 23,050(1.15)^x$

STEP 4 Use your exponential function model to predict the 2018 viewership, in number of households.

$2018 - 1970 = 48$, so the number of 3-year intervals since 1970, $x = 16$.

$f(16) = 23,050(1.15)^{16} \approx 215,693.2$

According to the exponential function model $f(x) = 23,050(1.15)^x$, approximately 215,693 households will view the Super Bowl in the year 2018.

YOU TRY IT! #2

A biologist places a single paramecium in a petri dish to observe the rate of population growth of this single-celled organism. The biologist's observations of the number of paramecia in the petri dish over time are recorded in the table below.

1-DAY INTERVAL, x	DAY OF EXPERIMENT	POPULATION, $f(x)$
0	1ST	1
1	2ND	2
2	3RD	5
3	4TH	10
4	5TH	22
5	6TH	44
6	7TH	87

Generate an exponential function model for the paramecium population. How many paramecia does your model predict will be in the petri dish on the tenth day of the experiment?

EXAMPLE 3

The population of Throckmorton County, Texas, in each census since 1950 is shown in the table below.

10-YEAR INTERVAL, x	YEAR	POPULATION, $f(x)$
0	1950	3,618
1	1960	2,767
2	1970	2,205
3	1980	2,053
4	1990	1,880
5	2000	1,850
6	2010	1,641

Data Source: U.S. Census Bureau

Use the data set to generate an exponential model. Use your model to predict the population of Throckmorton County, Texas, in the year 2020.

STEP 1 **Determine the finite differences in x-values and the ratios between successive values of $f(x)$.**

10-YEAR INTERVAL, x	YEAR	POPULATION, $f(x)$
0	1950	3,618
1	1960	2,767
2	1970	2,205
3	1980	2,053
4	1990	1,880
5	2000	1,850
6	2010	1,641

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

$\frac{y_n}{y_{n-1}} = \frac{2,767}{3,618} \approx 0.765$

$\frac{y_n}{y_{n-1}} = \frac{2,205}{2,767} \approx 0.797$

$\frac{y_n}{y_{n-1}} = \frac{2,053}{2,205} \approx 0.931$

$\frac{y_n}{y_{n-1}} = \frac{1,880}{2,053} \approx 0.916$

$\frac{y_n}{y_{n-1}} = \frac{1,850}{1,880} \approx 0.984$

$\frac{y_n}{y_{n-1}} = \frac{1,641}{1,850} \approx 0.887$

STEP 2 **Calculate the average of the ratios and use this value for the base, b, in your exponential function model.**

$$\frac{0.765 + 0.797 + 0.931 + 0.916 + 0.984 + 0.887}{6} \approx 0.88$$

STEP 3 **Using a, the initial population value from the table, and the calculated value for b, write an exponential function model.**

$f(x) = 3618(0.88)^x$

STEP 4 **Use your exponential function model to predict the 2020 population.**

$2020 - 1950 = 70$, so the number of 10-year intervals since 1950, $x = 7$.

$f(7) = 3618(0.88)^7 \approx 1,478.59$

According to the exponential function model $f(x) = 3618(0.88)^x$, the population of Throckmorton County, Texas, will be approximately 1,479 people in the year 2020.

YOU TRY IT! #3

Percentages of adults in the United States who smoke cigarettes on a daily basis are recorded in the table.

2-YEAR INTERVAL, x	YEAR	PERCENTAGE OF AMERICAN ADULTS, $f(x)$
0	1995	19.9
1	1997	19.1
2	1999	18.0
3	2001	17.4
4	2003	16.9
5	2005	15.3
6	2007	14.5

Data Source: Center for Disease Control (CDC)

Use the data set to generate an exponential model. Use your model to predict the percentage of adults in the United States who smoke cigarettes on a daily basis in the year 2015.

PRACTICE/HOMEWORK

For questions 1 and 2, determine whether a linear model or an exponential model would be most appropriate for the data. Explain how you made your decision.

1.

x	y
0	45
1	67.5
2	94.5
3	151.2
4	257.04
5	359.856

2.

x	y
0	209.5
1	184.6
2	159.6
3	134.4
4	109.6
5	84.5

For questions 3 - 5, calculate the average ratio between successive y-values.

3.

x	y
0	425.6
1	766.08
2	1225.73
3	2083.74
4	3750.73

4.

x	0	1	2	3	4	5	6
y	2300.6	1173.31	586.65	287.46	137.98	70.37	36.59

5.

x	0	1	2	3	4
y	1810.4	2172	2389.2	3105.96	3727.15

For questions 6 – 8, identify whether the data shows exponential growth or exponential decay. Then, determine an exponential function to model the situation.

SCIENCE

6. The population of gray squirrels in a local park has been recorded every year since 2005.

1-YEAR INTERVAL, x	YEAR	SQUIRREL POPULATION, y
0	2005	62
1	2006	87
2	2007	113
3	2008	170
4	2009	204
5	2010	265

FINANCE

7. Kristal noticed that her favorite painting in a museum has been increasing in value over the years. The changing value of the painting is shown in the table.

10-YEAR INTERVAL, x	YEAR	VALUE OF THE PAINTING, $f(x)$
0	1960	$2200
1	1970	$7500
2	1980	$25,000
3	1990	$86,000
4	2000	$292,000
5	2010	$992,000

SCIENCE

8. Mrs. Montgomery's class is doing an experiment with pennies. They empty a cup of pennies onto a table, and remove all the pennies that landed "heads-up." Then, they put the other pennies back in the cup, and repeat the process four more times.

TRIAL NUMBER, x	NUMBER OF PENNIES REMAINING, $fix)$
0	61
1	28
2	16
3	9
4	7
5	2

For questions 9 – 12 use the following situation.

FINANCE

Most cars decrease in value over time. The table below shows the value of Carla's car from the time of its purchase.

1-YEAR INTERVAL, x	YEAR	VALUE OF CAR, $f(x)$
0	2007	$29,870
1	2008	$24,180
2	2009	$20,480
3	2010	$17,420
4	2011	$14, 585
5	2012	$12,124

9. Use the data set to generate an exponential model.

10. What do the y-intercept and base from your function rule mean in context of the situation?

11. In what year will the car be worth about $5900?

12. Use your model to predict the value of Carla's car in the year 2020.

For questions 13 – 16 use the following situation.

FINANCE

Ella sells hair ribbons and decided to start marketing them on the internet hoping to increase her sales. The table shows the total number of ribbons she has sold.

NUMBER OF WEEKS SINCE MARKETING ON THE INTERNET, x	TOTAL NUMBER OF RIBBONS SOLD, $f(x)$
0	310
1	336
2	365
3	388
4	425
5	445
6	496

13. Use the data set to determine an exponential function that models the situation.

14. What is the y-intercept of this function, and what does it mean in context of the problem?

15. Use your function model to determine approximately how many weeks it will take to sell 1,000 ribbons.

16. Use your function model to predict how many ribbons Ella will sell in a year.

For questions 17 – 20 use the following situation.

SCIENCE

A biologist is recording the population of a certain bacteria in a petri dish. He determines the number of bacteria in the dish every 2 hours, as shown in the table below.

17. Does this data show exponential growth or exponential decay? Explain.

2-HR INTERVAL, x	NUMBER OF HOURS	NUMBER OF BACTERIA, $f(x)$
0	0	8
1	2	12
2	4	17
3	6	24
4	8	34
5	10	50

18. At some point, it becomes impossible to count all the bacteria, so an equation is necessary. Use the data set to generate an exponential model.

19. Use your exponential model to determine approximately how many bacteria will be in the petri dish after 1 day.

20. Approximately how many days will elapse before there are 451,000 bacteria?

For questions 21 – 24 use the following situation.

CRITICAL THINKING

Beth enjoys running for exercise. She has started a training plan that will gradually increase her weekly mileage as she prepares for a half-marathon. The table shows her training plan.

NUMBER OF WEEKS, x	MILES PER WEEK, $f(x)$
0	18
1	20
2	22
3	24
4	26.5
5	29
6	32
7	36

21. Using the data given above, generate an exponential function that models the situation.

22. What is the percent increase of her mileage from week to week on the plan?

23. According to your model, how much should Beth run in week 10 of her training plan?

24. If the training plan limits the mileage to 65 miles per week, when should she reach this goal?

Chapter 1 Mid-Chapter Review

1.1 Arithmetic and Geometric Sequences

1. For each sequence of numbers below, determine whether it is arithmetic, geometric or neither. Then, write a recursive rule and an explicit rule, if possible.

a) 128, 96, 72, 54, 40.5, …

b) 400, 300, 150, 75, 22.5, …

c) −5, −2.75, −0.5, 1.75, 4, …

2. Given each recursive rule, write the first 4 terms of the sequence. Indicate whether each sequence is arithmetic or geometric.

a) $a_1 = 200; a_n = a_{n-1} - 15$

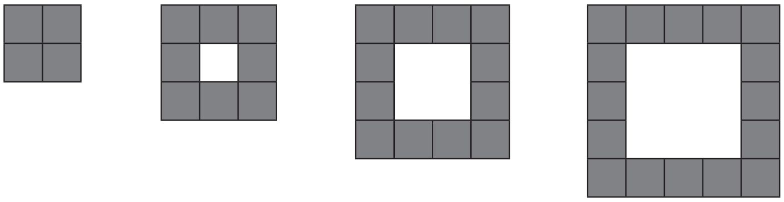
b) $a_1 = 12; a_n = \frac{2}{3}a_{\text{n-1}}$

3. Given each explicit rule, write the first 4 terms of the sequence. Indicate whether each sequence is arithmetic or geometric.

a) $a_n = 243\left(\frac{1}{3}\right)^{n-1}$

b) $a_n = 32 - 1.5n$

4. Write a rule to describe the number of shaded squares in each figure as a function of the figure number, n.



5. Evan's parents tell him he can choose one of two ways to earn his allowance for the coming year. He can either: a) receive \$1 in January, \$2 in February, \$4 in March and so on, doubling the amount he receives each month; or b) receive \$20 in January, \$25 in February, \$30 in March and so on, increasing his monthly allowance by \$5 each month.

How much will Evan's December allowance be under each plan? Write the explicit rule and then show how the answer is derived using the explicit rule.

1.2 Writing Linear Functions

Write an equation in $y = mx + b$ form for each linear function in problems 6 – 8 described below.

6. slope = 1.5, y-intercept = (0, 5)

7. slope = $\frac{-4}{3}$, contains the point (−3, 7)

8. contains the points (-2, -10), (4, -7), and (20, 1)

9. For each table, determine whether the relationship shows a linear function. If so, write the function.

a)

x	y
1	9
2	6
3	3
4	0
5	-3

b)

x	y
0	3.2
2	4.2
4	5.2
6	6.2
8	7.2

c)

x	y
1	0
2	4
3	8
4	16
5	24

10. Jerome weighed 210 pounds when he started on a diet. He plans to lose 2.5 pounds per week. Complete the table below to represent Jerome's weight over time and write a function to represent this situation.

# OF WEEKS	WEIGHT
0	
1	
2	
3	
4	

1.3 Modeling with Linear Functions

Use the following situation to answer the questions below.

Amanda was given a 5-week-old kitten for her birthday. The table below shows how the kitten's weight changed over the next 5 weeks.

TIME (WEEKS)	5	6	7	8	9	10
WEIGHT (GRAMS)	480	540	605	682	754	820

11. Write a function rule to model the situation.

12. What is the y-intercept from your function rule and what does it mean in the context of this situation?

13. What does the slope from your function rule mean in the context of this situation?

14. Use your model to predict the weight of Amanda's kitten when it is 22 weeks old.

15. In what week will the kitten weigh approximately 1300 grams?

1.4 Writing Exponential Functions

For questions 16 – 18, state whether the table represents an exponential function or not. If the table represents an exponential function, write the common ratio and the function relating the variables.

16.

x	y
0	240
1	180
2	135
3	101.25

17.

x	y
0	12
1	24
2	36
3	64

18.

x	y
1	750
2	900
3	1080
4	1296

For questions 19 – 20 use the situation below.

In its first 4 years of operation, a certain company increased its revenue by 2.5% each month.

NUMBER MONTHS, x	MONTHLY REVENUE, y
0	\$800.00
1	\$820.00
2	\$840.00
3	\$861.51
4	\$883.05

19. Write the function relating the variables.

20. If this pattern continues, what would be the company's monthly revenue at the end of 1 year?

1.5 Modeling with Exponential Functions

For questions 21 – 24 use the situation below.

Evan made a fresh pot of coffee then measured its temperature in 3-minute intervals over a period of 15 minutes, as shown in the table below.

NUMBER OF 3-MIN INTERVALS, x	NUMBER OF MINUTES	TEMPERATURE (F°), $f(x)$
0	0	180.2
1	3	172.5
2	6	163.8
3	9	155.6
4	12	147.9
5	15	140.8

21. Does this data show exponential growth or exponential decay? Explain.

22. Write an exponential model for this data.

23. Use your exponential model to determine the approximate temperature after $\frac{1}{2}$ hour.

24. Approximately how many minutes will elapse before the temperature is 80° F?

For questions 25 – 27 use the situation below.

The population of Harris County, TX over a 7-year period is shown in the table below, where x represents the number of years since 2000 and y represents the population in millions.

25. Using the data given above, generate an exponential function that models the situation.

26. According to your model, what is a reasonable estimate of the population in the year 2015?

27. According to your model, in what year is the population expected to reach 5 million?

YEARS SINCE 2000, x	POPULATION (IN MILLIONS), $f(x)$
0	3.4
1	3.46
2	3.52
3	3.58
4	3.64
5	3.71
6	3.78
7	3.84

MULTIPLE CHOICE

28. An auditorium has seating in which the front row has 25 seats, the 2nd row 28 seats, the 3rd row 31 seats, etc. If this pattern continues, how many seats will be in the 12th row?

A. 46
B. 52
C. 55
D. 58

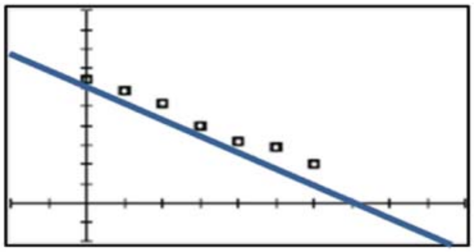
29. A t-shirt company charges a $20.00 set up fee plus $8.95 for each t-shirt that is screen-printed with a school's logo. Does this situation represent a linear relationship? If so, which function can be used to find the total cost, C, for purchasing and screen-printing t number of t-shirts?

A. yes, $C = 20t + 8.95$
B. yes, $C = 20 + 8.95t$
C. yes, $C = 20t + 8.95t$
D. Not a linear function

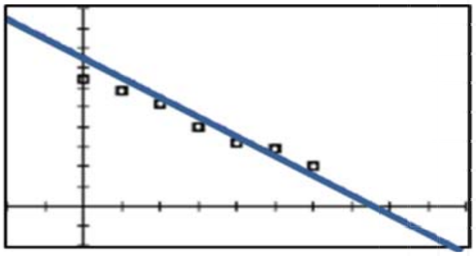
30. Which of the following graphs shows the best linear function model for the given data?

x	0	1	2	3	4	5	6
y	6.4	5.8	5.2	4	3.2	2.9	2

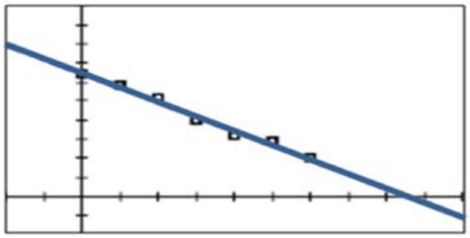
A.



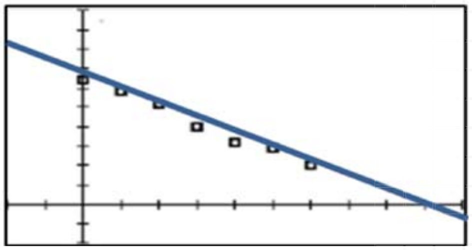
B.



C.



D.



31. Look at the table shown below.

x	y
0	64
1	80
2	100
3	125

Which of the following statements is NOT true about the table?

A. The function relating the variables is $y = 64(1.25)^x$.

B. The function is exponential.

C. The common ratio is 1.25.

D. The function represents exponential decay.

32. Which of the following functions best models the given data?

x	0	1	2	3	4	5	6
y	12.2	18.5	26.7	40	60.6	91.9	135.9

A. $y = 120 - 16.5x$

B. $y = 120 - 29x$

C. $y = 12.2(1.5)^x$

D. $y = 120(0.25)^x$

1.6 Writing Quadratic Functions

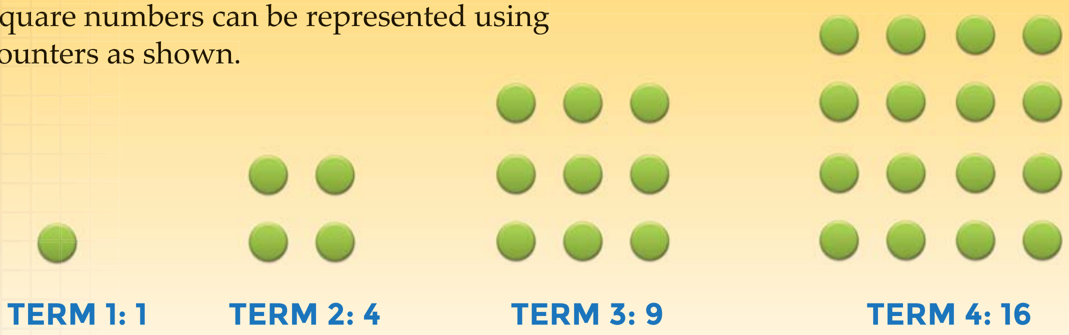
FOCUSING QUESTION What are the characteristics of a quadratic function?

LEARNING OUTCOMES

- I can determine patterns that identify a quadratic function from its related finite differences.
- I can determine the quadratic function from a table using finite differences, including any restrictions on the domain and range.
- I can use finite differences to determine a quadratic function that models a mathematical context.
- I can analyze patterns to connect the table to a function rule and communicate the quadratic pattern as a function rule.

ENGAGE

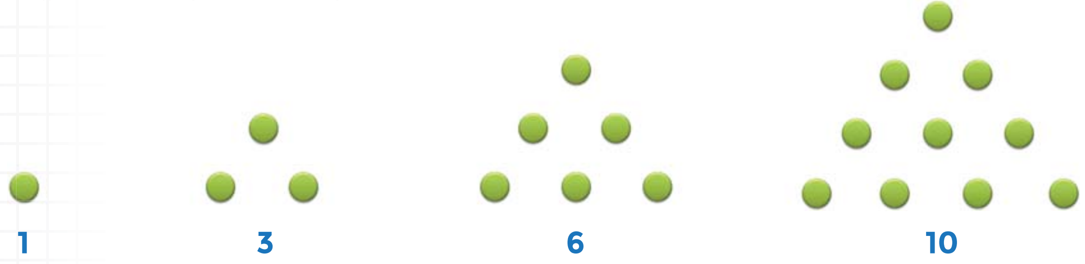
Square numbers can be represented using counters as shown.



What patterns do you see in the geometric arrangements of square numbers?

EXPLORE

A **figurative number**, sometimes called a **figurate number**, is a number that can be represented by a regular geometric arrangement of dots or other objects. For example, **triangular numbers** can be represented using arrangements of dots that are shaped like regular triangles.



Use counters to build a sequence of the first six triangular numbers. Record the numbers in a table like the one shown.

TERM NUMBER	TRIANGULAR NUMBER
1	1
2	
3	
4	
5	
6	

1. As you build the sequence, what patterns do you see in each successive term?

2. Does the data set follow a linear or an exponential function? Explain your reasoning.

3. What patterns do you see in the finite differences or the successive ratios?

4. Calculate the second finite difference. What do you notice?

5. The quadratic parent function is $y = x^2$. Generate a sequence with y-values for $\{x \mid x = 1, 2, 3, 4, 5, 6\}$.

A set of numbers, such as x-values or y-values, can be represented with braces using set notation. The set of whole numbers less than 10 is represented as $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If this set is a set of x-values, it can be written as $\{x \mid x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which is read "the set of all x such that x equals zero, one, two, …"

6. Calculate the second finite differences for the quadratic parent function. What do you notice?

x	y
1	1
2	4
3	9
4	16
5	25
6	36

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

7. What type of function do you think represents the relationship between the triangular number and the term number, or its position in the sequence?

REFLECT

- **In a linear function, the first finite differences are constant. What is true about the finite differences for a quadratic function?**
- **A linear function contains a polynomial with degree one ($mx + b$) and a quadratic function contains a polynomial with degree two ($ax^2 + bx + c$). What relationship is there between the degree of the polynomial and the level of finite differences that are constant?**

EXPLAIN

In a linear function, the first finite differences, or the difference between consecutive values of the dependent variable, are constant. But for a quadratic function, the first finite differences are not constant. They do, however, have a pattern in that they increase or decrease by the same number. As a result, the second finite differences, or the differences between the first finite differences, are constant.

There are many forms of a quadratic function. **Polynomial form**, also called **standard form**, expresses the function as a polynomial with exponents in decreasing order.

$$f(x) = ax^2 + bx + c$$

In standard form, a, b, and c are rational numbers.

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Let's look more closely at a quadratic function. The table below shows the relationship between x and $f(x)$ in a quadratic function written in polynomial or standard, $f(x) = ax^2 + bx + c$.

x	PROCESS	$y = f(x)$
0	$a(0)^2 + b(0) + c$	c
1	$a(1)^2 + b(1) + c$	$a + b + c$
2	$a(2)^2 + b(2) + c$	$4a + 2b + c$
3	$a(3)^2 + b(3) + c$	$9a + 3b + c$
4	$a(4)^2 + b(4) + c$	$16a + 4b + c$
5	$a(5)^2 + b(5) + c$	$25a + 5b + c$

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta y = (a + b + c) - c = a + b$

$\Delta y = (4a + 2b + c) - (a + b + c) = 3a + b$

$\Delta y = (9a + 3b + c) - (4a + 2b + c) = 5a + b$

$\Delta y = (16a + 4b + c) - (9a + 3b + c) = 7a + b$

$\Delta y = (25a + 5b + c) - (16a + 4b + c) = 9a + b$

The first differences are not constant, but there is a pattern as the differences increase from $a + b$ to $3a + b$, from $3a + b$ to $5a + b$, and so on. So let's look at the second differences. When you look at the second differences, three patterns emerge.

x	$y = f(x)$
0	c
1	$a + b + c$
2	$4a + 2b + c$
3	$9a + 3b + c$
4	$16a + 4b + c$
5	$25a + 5b + c$

$$\Delta y = (a + b + c) - c = a + b$$
$$\Delta y = (4a + 2b + c) - (a + b + c) = 3a + b$$
$$\Delta y = (9a + 3b + c) - (4a + 2b + c) = 5a + b$$
$$\Delta y = (16a + 4b + c) - (9a + 3b + c) = 7a + b$$
$$\Delta y = (25a + 5b + c) - (16a + 4b + c) = 9a + b$$

$$\Delta^2 y = (3a + b) - (a + b) = 2a$$
$$\Delta^2 y = (5a + b) - (3a + b) = 2a$$
$$\Delta^2 y = (7a + b) - (5a + b) = 2a$$
$$\Delta^2 y = (9a + b) - (7a + b) = 2a$$

You can use these three patterns to determine the quadratic function from the table of data.

- The value of c is the y-coordinate of the y-intercept, $(0, c)$.
- The second difference is equal to $2a$.
- The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b$.

FINITE DIFFERENCES AND QUADRATIC FUNCTIONS

In a quadratic function, the second differences between successive y-values are constant if the differences between successive x-values, Δx, are also constant.

If the second differences between consecutive y-values in a table of values are constant, then the values represent a quadratic function.

The formulas for finding the values of a, b, and c to write the quadratic function only work when $\Delta x = 1$. When $\Delta x \neq 1$, there are other formulas that can be used to determine the values of a, b, and c for the quadratic function.

EXAMPLE 1

What type of function would best model the data set below? Justify your answer.

x	y
0	0
1	1
2	6
3	15
4	28

STEP 1 **Determine whether or not the set of data represents a linear function.**

x	y
0	0
1	1
2	6
3	15
4	28

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta y = 1 - 0 = 1$

$\Delta y = 6 - 1 = 5$

$\Delta y = 15 - 6 = 9$

$\Delta y = 28 - 15 = 13$

The differences in x, Δx, are all 1, so they are constant.

The differences in y, Δy, are not all the same, so they are not constant.

Therefore, the set of data does not represent a linear function because the first finite differences are not constant.

STEP 2 Determine whether or not the set of data represents an exponential function.

x	y
0	0
1	1
2	6
3	15
4	28

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\frac{y_1}{y_0} = \frac{1}{0}$ (undefined)

$\frac{y_2}{y_1} = \frac{6}{1} = 6$

$\frac{y_3}{y_2} = \frac{15}{6} = 2.5$

$\frac{y_4}{y_3} = \frac{28}{15} = 1.8666...$

The data set is not exponential because the successive ratios are not constant.

STEP 3 Determine whether or not the set of data represents a quadratic function.

x	y
0	0
1	1
2	6
3	15
4	28

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta y = 1 - 0 = 1$

$\Delta y = 6 - 1 = 5$

$\Delta y = 15 - 6 = 9$

$\Delta y = 28 - 15 = 13$

$\Delta^2 y = 5 - 1 = 4$

$\Delta^2 y = 9 - 5 = 4$

$\Delta^2 y = 13 - 9 = 4$

The second finite differences are all 4, so the set of data represents a quadratic function.

YOU TRY IT! #1

Determine if the function rule for the set of data is linear, exponential, or quadratic.

x	y
0	0
1	1
2	7
3	19
4	37

EXAMPLE 2

Determine the function rule for the set of triangular numbers shown below.

x	y
0	0
1	1
2	3
3	6
4	10

STEP 1 Determine the finite differences between successive x-values and successive y-values.

x	y
0	0
1	1
2	3
3	6
4	10

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta y = 1 - 0 = 1$

$\Delta y = 3 - 1 = 2$

$\Delta y = 6 - 3 = 3$

$\Delta y = 10 - 6 = 4$

STEP 2 Determine whether or not the differences are constant.

The differences in x, Δx, are all 1, so they are constant.

The differences in y, Δy, are not constant.

STEP 3 Determine whether or not the second finite differences in successive y-values are constant.

x	y
0	0
1	1
2	3
3	6
4	10

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta y = 1 - 0 = 1$

$\Delta y = 3 - 1 = 2$

$\Delta y = 6 - 3 = 3$

$\Delta y = 10 - 6 = 4$

$\Delta^2 y = 2 - 1 = 1$

$\Delta^2 y = 3 - 2 = 1$

$\Delta^2 y = 4 - 3 = 1$

The second differences are all equal to 1 and are constant.

STEP 4 **Calculate a, b, and c for the quadratic function $f(x) = ax^2 + bx + c$. For $x = 0$, $y = 0$. So $c = 0$.**

The second finite difference is $2a$, so $2a = 1$ and $a = \frac{1}{2}$.

The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b$, so $a + b = 1$. Since a = $\frac{1}{2}$, b must also equal $\frac{1}{2}$.

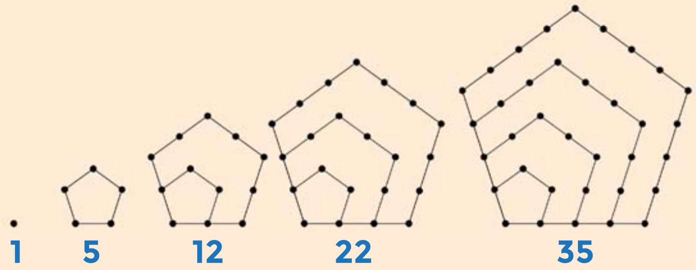
STEP 5 **Write the function with the values for a, b, and c:**

$$f(x) = \frac{1}{2}x^2 + \frac{1}{2}x + 0 \text{ or } f(x) = \frac{x^2 + x}{2}$$

YOU TRY IT! #2

Determine the function rule for the set of pentagonal numbers using the values in the table. Use mental math to calculate the finite differences.

x	y
0	0
1	1
2	5
3	12
4	22



EXAMPLE 3

Write a quadratic function where the second finite difference is 4, the y-intercept is (0,1), and $a + b$ is 5.

STEP 1 **Determine the values of a, b, and c.**

The second finite difference is 4. So $2a = 4$, and $a = 2$.
The y-value of the y-intercept, c, is 1.
Since $a + b = 5$ and $a = 2$, then $b = 3$.

STEP 2 Write a quadratic function in standard form: $ax^2 + bx + c$.

$f(x) = 2x^2 + 3x + 1$

YOU TRY IT! #3

For the data set below, write a function relating the variables.

x	y
1	1
2	9
3	23
4	43
5	69

PRACTICE/HOMEWORK

For questions 1 – 8, use finite differences and mental math, as appropriate, to determine if the data sets shown in the tables below represent a linear, exponential, quadratic, or other type of function.

1.

x	$y = f(x)$
1	5
2	11
3	21
4	35
5	53

2.

x	$y = f(x)$
1	5
2	11
3	17
4	23
5	29

3.

x	$y = f(x)$
1	5
2	9
3	16
4	29
5	52

4.

x	$y = f(x)$
1	5
2	14
3	29
4	50
5	77

5.

x	$y = f(x)$
1	5
2	12
3	31
4	68
5	129

6.

x	$y = f(x)$
1	5
2	8
3	13
4	20
5	29

7.

x	$y = f(x)$
1	5
2	11
3	24
4	53
5	117

8.

x	$y = f(x)$
1	5
2	9
3	13
4	17
5	21

For questions 9 – 12, the data sets shown in the tables represent quadratic functions. Use finite differences to determine the values of a, b, and c and then write the function in standard form.

9.

x	$y = f(x)$
0	7
1	10
2	19
3	34

10.

x	$y = f(x)$
0	3
1	6
2	13
3	24

11.

x	$y = f(x)$
0	-1
1	5
2	19
3	41

12.

x	$y = f(x)$
0	-6
1	-1
2	14
3	39

For questions 13 – 16, the data sets shown in the tables represent quadratic functions. Use finite differences to determine f(0), the values of a, b, and c and then write the function in standard form.

13.

x	$y = f(x)$
0	?
1	-1
2	5
3	13
4	23

14.

x	$y = f(x)$
0	?
1	3
2	16
3	41
4	78

15.

x	$y = f(x)$
0	?
1	-9
2	-8
3	-1
4	12

16.

x	$y = f(x)$
0	?
1	7
2	22
3	45
4	76

For questions 17 – 20 use the situation below.

CRITICAL THINKING

Toothpicks were used to create the pattern below.

1

2

3

17. Relate the length of one side of the figure, x, to the area of the figure, y, by completing the table below. The first row has been completed for you.

LENGTH x	AREA y
1	1
2	
3	

18. Write the function relating the variables in problem 17.

19. If the pattern continues, what would be the area of a figure with a side length of 7?

20. Relate the figure number, x, to the total number of toothpicks needed to create the figure, y, by completing the table below. The first row has been completed for you.

FIGURE NUMBER x	TOTAL TOOTHPICKS y
1	4
2	
3	

21. Write the function relating the variables in problem 20.

22. If the pattern continues, how many toothpicks would be needed to create Figure 5?

For questions 23 – 24 use the situation below.

SCIENCE

GRAVITY EXPERIMENT

An experiment is conducted by dropping an object from a height of 150 feet and measuring the distance it has fallen at 1-second intervals. Identical objects were used to perform the experiment on Venus, Earth, and Mars. The tables below show the results of each experiment.

VENUS	
TIME (SEC) x	DISTANCE (FEET) y
0	0
1	14.8
2	59.2
3	133.2

EARTH	
TIME (SEC) x	DISTANCE (FEET) y
0	0
1	16
2	64
3	144

MARS	
TIME (SEC) x	DISTANCE (FEET) y
0	0
1	6.2
2	24.8
3	55.8

23. Determine if each table represents a linear, exponential, or quadratic function.
Venus: Earth: Mars:

24. Write a function relating the variables in each of the tables above.
Venus:
Earth:
Mars:

1.7 Modeling with Quadratic Functions

FOCUSING QUESTION How can you use finite differences to construct a quadratic model for a data set?

LEARNING OUTCOMES

- I can use finite differences or common ratios to classify a function as linear, quadratic, or exponential when I am given a table of values.
- I can use finite differences to write a quadratic function that describes a data set.
- I can apply mathematics to problems that I see in everyday life, in society, and in the workplace.

ENGAGE

Mrs. Hernandez wants to construct a rectangular sandbox for her niece and nephew. She has 36 feet of lumber to use as the border. What are some possible dimensions that Mrs. Hernandez could use to construct the sandbox?

EXPLORE

Use color tiles to build rectangles that represent a sandbox with a perimeter of 36. Recall that the area of a rectangle can be found using the formula $A = lw$ and the perimeter of a rectangle can be found using the formula $P = 2(l + w)$. Record the dimensions and the area of each rectangle in a table like the one shown. When you make your table, list the rectangles with widths in order from least to greatest.

WIDTH (IN.)	LENGTH (IN.)	AREA (SQ. IN.)
3	15	45

1. What do you think a scatterplot of area versus width for your set of rectangles would look like?

2. Sketch a scatterplot of area versus width for your set of rectangles.

3. Based on the shape of your graph, what type of function best models the data?

4. Calculate the finite differences between the width and the area. Do the data appear to be linear or quadratic? How do you know?

5. Use the patterns in the finite differences to write a function rule that describes the data set.

6. Use a graphing calculator to graph the function rule over the scatterplot. What do you notice about the function rule and the scatterplot?

7. Compare the domain and range of the data set and the domain and range of the function rule. How are they alike? How are they different?

8. What dimensions will give Mrs. Hernandez a sandbox with the greatest area? Use the table and graph to justify your answer.

9. Are the x-intercepts of the function included in your data set? Why or why not?

10. The perimeter of a rectangle is found using the formula $P = 2l + 2w$. If you know the perimeter is 36 feet, how are the length and width related?

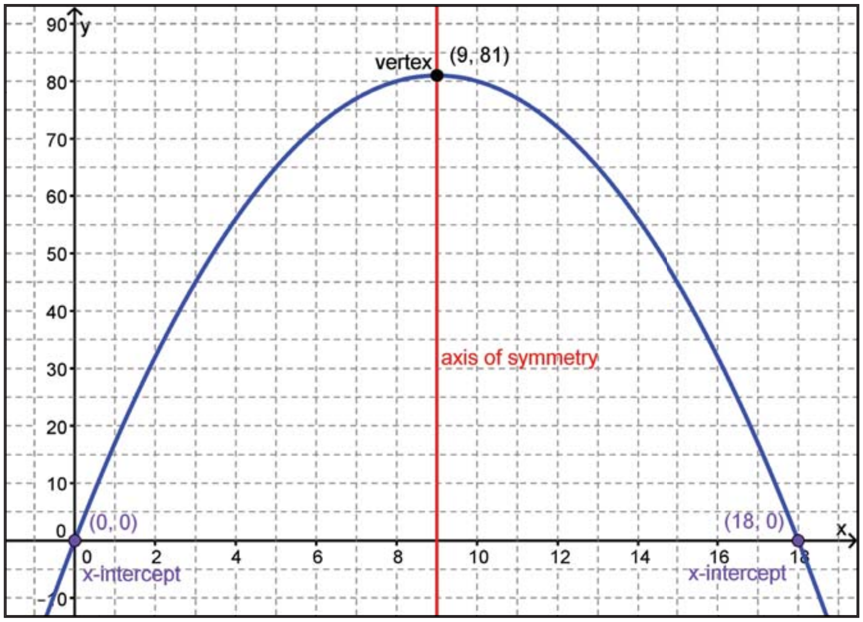
11. Use the relationship between the length and width of a rectangle with a fixed perimeter to write an equation for the area of the rectangle, $A = lw$. How does this equation compare with the function rule that you generated from finite differences?

REFLECT

- **How can you determine a quadratic function model for a data set?**
- **How do the parameters obtained from finite differences relate to the data set being modeled?**

EXPLAIN

Quadratic function models can be used to represent sets of mathematical and real-world data. A quadratic function has several key attributes that are important to consider when using and interpreting models that are based on quadratic functions. The graph of the quadratic function, which is a parabola, helps to explain how these attributes relate to the quadratic function model.



The **vertex of the parabola** represents the data values that generate a minimum or maximum value. In the case of the sandbox problem, the vertex reveals the width, or x-coordinate, that generates the maximum area, or y-coordinate. No other x-value in this model will generate a function value greater than the y-value of the vertex.

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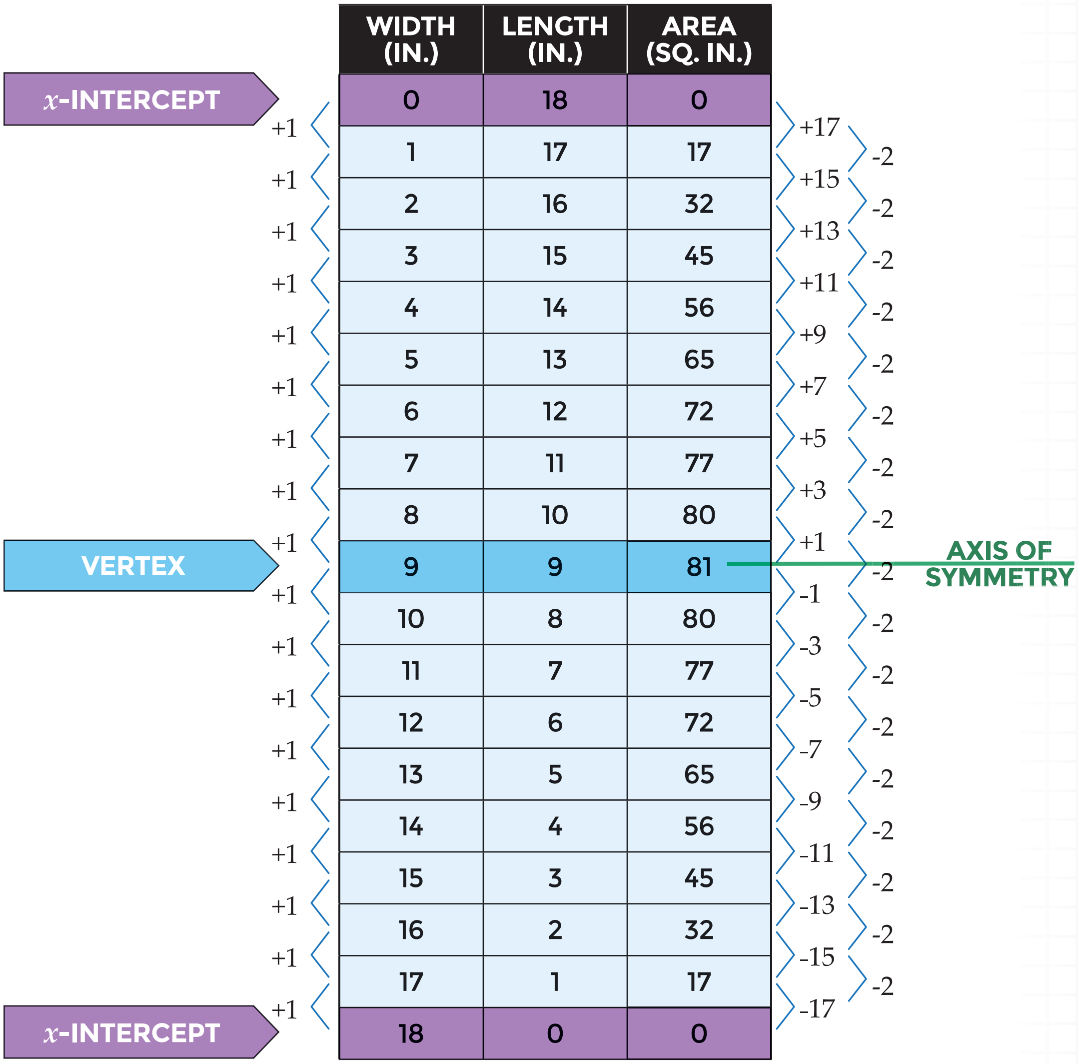
The **axis of symmetry** is a vertical line that represents the dividing line between the part of the parabola with increasing y-values and the part of the parabola with decreasing y-values. The axis of symmetry passes through the vertex.

The **x-intercepts** represent the points with function values that are equal to 0.

In the sandbox problem, you can identify the vertex, axis of symmetry, and x-intercepts from the table of values.

- The **vertex** is the row containing the greatest function value, or the greatest area.
- The **axis of symmetry** is represented by the x-value in the row where the function values change from increasing from row to row to decreasing from row to row.
- The **x-intercept** is a row in which the function value is 0.



WIDTH (IN.)	LENGTH (IN.)	AREA (SQ. IN.)
0	18	0
1	17	17
2	16	32
3	15	45
4	14	56
5	13	65
6	12	72
7	11	77
8	10	80
9	9	81
10	8	80
11	7	77
12	6	72
13	5	65
14	4	56
15	3	45
16	2	32
17	1	17
18	0	0

You can use the values of the key attributes of a quadratic function in order to interpret the model. In the sandbox problem, the function $f(x) = -x^2 + 18x$ models the data set. The vertex of this function, (9, 81), reveals that a width of 9 feet generates a maximum area of 81 square feet. All x-values to the left of the vertex represent the part of the function where the area increases as the width increases. All x-values to the right of the vertex represent the part of the function where the area decreases as the width increases.

In the sandbox problem, the x-intercepts do not make sense. An x-intercept of (18, 0) means that when the width of the sandbox is 18 feet, the area of the sandbox is 0 square feet. A rectangle cannot have an area of 0 square feet. The x-intercepts represent domain restrictions on the function model when it is applied to the situation.

	DOMAIN	RANGE
FUNCTION	$x \in \mathbb{R}$ (ALL REAL NUMBERS)	$y \leq 81$
SITUATION	$0 < x < 18$ (INCLUDES WHOLE AND REAL NUMBERS)	$0 < y \leq 81$

In the real world, you cannot have lengths that are negative or 0; they must be positive numbers. When using color tiles to represent the situation, you are further limiting the domain to only whole numbers. But in reality, Mrs. Hernandez could create a sandbox with fractional side lengths, such as 3.5 feet by 14.5 feet. Such a sandbox has a perimeter of 36 feet and meets the criteria of the problem.

MODELING WITH QUADRATIC FUNCTIONS

Real-world data rarely follows exact patterns, but you can use patterns in data to look for trends. If the data set has a constant or approximately constant second finite difference, then a quadratic function model may be appropriate for the data set.

The domain and range for the situation may be a subset of the domain and range of the quadratic function model.

EXAMPLE 1

A ball is thrown from the top of a building. The table below shows the height of a ball above the ground at one-second intervals. Determine whether the set of data represents a linear, quadratic, or exponential function.

TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$
0	100
1	105.1
2	100.4
3	85.9
4	61.6
5	27.5

STEP 1 Determine the finite differences in values of x and values of $f(x)$.

TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$
0	100
1	105.1
2	100.4
3	85.9
4	61.6
5	27.5

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta f(x) = 105.1 - 100 = 5.1$

$\Delta f(x) = 100.4 - 105.1 = -4.7$

$\Delta f(x) = 85.9 - 100.4 = -14.5$

$\Delta f(x) = 61.6 - 85.9 = -24.3$

$\Delta f(x) = 27.5 - 61.6 = -34.1$

STEP 2 Determine the ratios between successive values of $f(x)$.

TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$
0	100
1	105.1
2	100.4
3	85.9
4	61.6
5	27.5

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\frac{y_n}{y_{n\text{-}1}} = \frac{105.1}{100} \approx 1.051$

$\frac{y_n}{y_{n\text{-}1}} = \frac{100.4}{105.1} \approx 0.955$

$\frac{y_n}{y_{n\text{-}1}} = \frac{85.9}{100.4} \approx 0.856$

$\frac{y_n}{y_{n\text{-}1}} = \frac{61.6}{85.9} \approx 0.717$

$\frac{y_n}{y_{n\text{-}1}} = \frac{27.5}{61.6} \approx 0.446$

STEP 3 Determine the second finite differences in the $f(x)$ values.

TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$
0	100
1	105.1
2	100.4
3	85.9
4	61.6
5	27.5

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

+5.1

-4.7

-14.5

-24.3

-34.1

-9.8

-9.8

-9.8

-9.8

STEP 4 Determine whether the finite differences or the ratios between successive values of $f(x)$ are approximately constant.

- The first finite differences range in value from 5.1 to –34.1. This is a wide range, so the first finite differences are not approximately constant.
- The ratios between successive values of $f(x)$ range from 0.446 to 1.051. This is also a wide range, so the ratios between successive values of $f(x)$ are not approximately constant.
- The second finite differences are all –9.8 and are constant.

The set of data represents a quadratic function, rather than a linear or exponential function, because the differences in x are constant and the second finite differences in f(x) are constant.

YOU TRY IT! #1

A softball pitcher throws a ball to her catcher. The ball's path is tracked in the table.

HORIZONTAL DISTANCE FROM PITCHER IN FEET, x	VERTICAL HEIGHT IN FEET, $f(x)$
0	2
5	5.6
10	8.4
15	10.4
20	11.6
25	12

Determine whether the relationship is linear, exponential, or quadratic.

EXAMPLE 2

The total amount in a savings account is shown in the table. Determine whether the interest that is being earned in the savings account follows a linear, quadratic, or exponential function.

1-YEAR INTERVAL, x	INTEREST IN DOLLARS, $f(x)$
0	500
1	530
2	561.80
3	595.51
4	631.24

STEP 1 Determine the finite differences in values of x and values of $f(x)$.

1-YEAR INTERVAL, x	INTEREST IN DOLLARS, $f(x)$
0	500
1	530
2	561.80
3	595.51
4	631.24

$\Delta f(x) = 530 - 500 = 30$

$\Delta f(x) = 561.80 - 530 = 31.80$

$\Delta f(x) = 595.51 - 561.80 = 33.71$

$\Delta f(x) = 631.24 - 595.51 = 35.73$

The differences in the x-values are constant and the first finite differences in the values for $f(x)$ range from 30 to 35.73.

STEP 2 Determine the ratios between successive values of $f(x)$.

1-YEAR INTERVAL, x	INTEREST IN DOLLARS, $f(x)$
0	500
1	530
2	561.80
3	595.51
4	631.24

$\frac{y_n}{y_{n\text{-}1}} = \frac{530}{500} \approx 1.06$

$\frac{y_n}{y_{n\text{-}1}} = \frac{561.80}{530} \approx 1.06$

$\frac{y_n}{y_{n\text{-}1}} = \frac{595.51}{561.80} \approx 1.061$

$\frac{y_n}{y_{n\text{-}1}} = \frac{631.24}{595.51} \approx 1.059$

The ratios between successive values of f(x) are approximately 1.06. These values are all close together, so the ratios between successive values of f(x) are approximately constant, and the data set represents an exponential function.

YOU TRY IT! #2

The total amount in a savings account is shown in the table. Determine whether the interest that is being earned in the savings account follows a linear, quadratic, or exponential function.

1-YEAR INTERVAL, x	TOTAL AMOUNT IN DOLLARS, $f(x)$
0	750
1	780
2	810
3	840
4	870

EXAMPLE 3

Possum Kingdom Lake in Palo Pinto County, Texas, was the setting for a world-class cliff diving in 2014. The champion diver's approximate position during the dive is recorded in the table.

DISTANCE AWAY FROM THE CLIFF IN METERS, x	HEIGHT ABOVE THE WATER IN METERS, $f(x)$
0	27
1	28.1
2	27.4
3	24.8
4	20.0
5	12.9

Data Source: Redbullcliffdiving.com

Use the data set to generate a quadratic function that best models the data.

Use the table to estimate the height of the cliff, the height of the diver at his highest point, and his distance from the cliff when he entered the water.

STEP 1 **Determine the finite differences in x-values and the second finite differences in the values of $f(x)$.**

	DISTANCE AWAY FROM THE CLIFF IN METERS, x	HEIGHT ABOVE THE WATER IN METERS, $f(x)$		
	0	27		
$\Delta x = 1 - 0 = 1$			+1.1	
	1	28.1		-1.8
$\Delta x = 2 - 1 = 1$			-0.7	
	2	27.4		-1.9
$\Delta x = 3 - 2 = 1$			-2.6	
	3	24.8		-2.2
$\Delta x = 4 - 3 = 1$			-4.8	
	4	20.0		-2.3
$\Delta x = 5 - 4 = 1$			-7.1	
	5	12.9		

STEP 2 **Calculate the average of the second finite differences and use this value to determine a in your quadratic function model, $f(x) = ax^2 + bx + c$.**

$$2a = \frac{-1.8 -1.9 -2.2 - 2.3}{4} = -2.05$$

So $a = -1.025$

STEP 3 **Calculate the value of b.**

The difference between the values of $f(x)$ for $x = 0$ and 1 is $(a + b)$.

$$\begin{aligned} a + b &= 1.1 \\ (-1.025) + b &= 1.1 \\ b &= 2.125 \end{aligned}$$

STEP 4 Determine the value of c.

The value of $f(0) = c$. $f(0) = 27$, so $c = 27$.

STEP 5 Substitute the values of a, b, and c into the general form to determine the function model.

The quadratic function model is $f(x) = -1.025x^2 + 2.125x + 27$.

Using the table, the top of the cliff must have been about 27 meters above the water. The highest point of the dive was a little more than 28 meters, and the diver entered the water at a distance of a little more than 6 meters from the cliff.

YOU TRY IT! #3

A study compared the speed x (in miles per hour) and the average fuel economy $f(x)$ (in miles per gallon) for cars. The results in 10 mile per hour increments over 20 mph are shown in the table.

10-MILE PER HOUR INTERVAL, x	MILES PER HOUR	GASOLINE USAGE IN MILES PER GALLON, $f(x)$
0	20	24.5
1	30	28.0
2	40	30.0
3	50	30.2
4	60	28.8
5	70	25.8

Use the data set to generate a quadratic model. Use your model to predict the fuel economy at 80 miles per hour.

PRACTICE/HOMEWORK

For questions 1-6 determine whether the set of data represents a linear, quadratic, or exponential function.

1.

x	$y = f(x)$
1	7
2	16
3	27
4	40
5	55

2.

x	$y = f(x)$
1	-4
2	-1
3	2
4	5
5	8

3.

x	$y = f(x)$
1	-13
2	-28
3	-45
4	-64
5	-85

4.

x	$y = f(x)$
1	2
2	4
3	8
4	16
5	32

5.

x	$y = f(x)$
1	-4
2	-6
3	-6
4	-4
5	0

6.

x	$y = f(x)$
1	0.2
2	0.04
3	0.008
4	0.0016
5	0.00032

For questions 7-12 use the data set to generate a quadratic function that best models the data.

7.

x	$y = f(x)$
1	3
2	12
3	27
4	48
5	75

8.

x	$y = f(x)$
1	2
2	2
3	0
4	-4
5	-10

9.

x	$y = f(x)$
1	-12
2	-20
3	-24
4	-24
5	-20

10.

x	$y = f(x)$
1	8.5
2	18
3	28.5
4	40
5	52.5

11.

x	$y = f(x)$
1	1
2	-8
3	-23
4	-44
5	-71

12.

x	$y = f(x)$
1	6
2	28
3	58
4	96
5	142

For questions 13 and 14, use the following information.

SCIENCE

The Texas Department of Public Safety can use the length of skid marks to help determine the speed of a vehicle before the brakes were applied. The quadratic function that best models the data is $f(x) = \frac{x^2}{24}$ where x represents the speed of the vehicle and $f(x)$ is the length of the skid mark. The speeds of a vehicle and the length of the corresponding skid marks are shown in the table below.

SPEED OF A VEHICLE IN MILES PER HOURS, x	DISTANCE OF THE SKID IN FEET, $f(x)$
30	37.5
36	54
42	73.5
48	96
54	121.5
60	150

13. Use the table of data to determine the length of a skid mark of a vehicle that was traveling at a speed of 72 miles when it applied the brakes.

14. Use the table of data to determine how fast a vehicle was traveling if the length of the skid mark was 24 feet.

For questions 15 - 17, use the following information.

SCIENCE

A ball is thrown upward with an initial velocity of 35 meters per second. The position of the ball over time is recorded in the table below.

15. Use the data in the table to generate a quadratic function that models the data.

16. Use the data in the table to find the height of the ball after 7 seconds.

17. Use the data in the table to determine after how many seconds the ball will be 30 meters high.

TIME IN SECONDS, x	DISTANCE FROM THE GROUND IN METERS, $f(x)$
0	0
1	30
2	50
3	60
4	60
5	50

For questions 18 - 20, use the following information.

GEOMETRY

Judy wants to construct a rectangular pen for her puppy, but only has 56 feet of fencing to use for the pen. The table below shows the width, length, and area of different size pens.

WIDTH (FT)	LENGTH (FT)	AREA (SQ. FT.)
10	18	180
11	17	187
12	16	192
13	15	195
14	14	196
15	13	195
16	12	192

18. Use the data in the table to generate a quadratic function that models the data.

19. Use the data in the table to determine the dimensions that would create a pen with an area of 160 ft^2.

20. Use the data in the table to determine the area of a pen where one of the dimensions measures 20 feet.

Writing Cubic Functions

1.8

FOCUSING QUESTION What are the characteristics of a cubic function?

LEARNING OUTCOMES

- I can determine patterns that identify a cubic function from its related finite differences.
- I can determine the cubic function from a table using finite differences, including any restrictions on the domain and range.
- I can use finite differences to determine a cubic function that models a mathematical context.
- I can analyze patterns to connect the table to a function rule and communicate the cubic pattern as a function rule.

ENGAGE

A tomato sauce can is in the shape of a cylinder and the diameter of the base is equal to the height of the can. Generate a sequence showing the volumes of a series of cans with a radius of 1 inch, 2 inches, 3 inches, and 4 inches. What patterns do you notice in the sequence?

EXPLORE

The volume of a prism is found using the area of the base and the height of the prism, $V = Bh$. If the prism is a rectangular prism, then the base is a rectangle and its area is the product of the length and width of the rectangle, $A = lw$. Combining these formulas generates a formula you can use to determine the volume of a rectangular prims, $V = lwh$.

Use cubes to build the first two terms in a sequence of rectangular prisms. For this sequence, the term number is the length of the prism, the width of the prism is double the term number, and the height of the prism is triple the term number.

1. Sketch the first two terms that you built with the cubes.

2. Complete a table like the one shown using the relationships among the dimensions of the prism (length = x, width = $2x$, and height = $3x$).

TERM NUMBER	PROCESS	VOLUME
1	1(2)(3)	6
2		
3		
4		
5		
6		

3. Does the data set follow a linear or an exponential function? Explain your reasoning.

4. Calculate the second finite differences. What do you notice?

5. Calculate the third finite differences. What do you notice?

6. Use the relationships among the dimensions that you were originally given to calculate the volume of a rectangular prism with a length of x units. What type of function does this appear to be?

REFLECT

- **A cubic function is a function of the form $f(x) = ax^3 + bx^2 + cx + d$. What is the degree of this function (i.e., the power of the greatest exponent)?**
- **A linear function contains a polynomial with degree one ($mx + b$) and a quadratic function contains a polynomial with degree two ($ax^2 + bx + c$). A cubic function contains a polynomial with a degree three ($ax^3 + bx^2 + cx + d$). What relationship is there between the degree of the polynomial and the level of finite differences that are constant?**

EXPLAIN

In a linear function, the first finite differences, or the difference between consecutive values of the dependent variable, are constant. For a quadratic function, the second finite differences are constant. In a cubic function, the third finite differences are constant.

Let's look more closely at a cubic function. The table below shows the relationship between x and $f(x)$. In a cubic function written in polynomial or standard, $f(x) = ax^3 + bx^2 + cx + d$.

There are many forms of a cubic function. Polynomial form, also called standard form, expresses a function as a polynomial with exponents in decreasing order.

$$f(x) = ax^3 + bx^2 + cx + d$$

In standard form, a, b, c, and d are rational numbers.

x	PROCESS	$y = f(x)$
0	$a(0)^3 + b(0)^2 + c(0) + d$	d
1	$a(1)^3 + b(1)^2 + c(1) + d$	$a + b + c + d$
2	$a(2)^3 + b(2)^2 + c(2) + d$	$8a + 4b + 2c + d$
3	$a(3)^3 + b(3)^2 + c(3) + d$	$27a + 9b + 3c + d$
4	$a(4)^3 + b(4)^2 + c(4) + d$	$64a + 16b + 4c + d$
5	$a(5)^3 + b(5)^2 + c(5) + d$	$125a + 25b + 5c + d$

$\Delta x = 1 - 0 = 1$
$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$

$\Delta y = a + b + c$
$\Delta y = 7a + 3b + c$
$\Delta y = 19a + 5b + c$
$\Delta y = 37a + 7b + c$
$\Delta y = 61a + 9b + c$

The first differences are not constant, so let's look at the second differences.

x	PROCESS	$y = f(x)$
0	$a(0)^3 + b(0)^2 + c(0) + d$	d
1	$a(1)^3 + b(1)^2 + c(1) + d$	$a + b + c + d$
2	$a(2)^3 + b(2)^2 + c(2) + d$	$8a + 4b + 2c + d$
3	$a(3)^3 + b(3)^2 + c(3) + d$	$27a + 9b + 3c + d$
4	$a(4)^3 + b(4)^2 + c(4) + d$	$64a + 16b + 4c + d$
5	$a(5)^3 + b(5)^2 + c(5) + d$	$125a + 25b + 5c + d$

$\Delta x = 1 - 0 = 1$
$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$

$\Delta y = a + b + c$
$\Delta y = 7a + 3b + c$
$\Delta y = 19a + 5b + c$
$\Delta y = 37a + 7b + c$
$\Delta y = 61a + 9b + c$

$6a + 2b$
$12a + 2b$
$18a + 2b$
$24a + 2b$

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The second differences are not constant, either. But, you can see some patterns emerging. Notice that every time you take another round of finite differences, the last constant term drops off because it is subtracted out. For the second differences, the coefficients of the a term are multiples of 6. Also, each second difference has the same b term, $2b$. Let's calculate the third differences.

x	PROCESS	$y = f(x)$
0	$a(0)^3 + b(0)^2 + c(0) + d$	d
1	$a(1)^3 + b(1)^2 + c(1) + d$	$a + b + c + d$
2	$a(2)^3 + b(2)^2 + c(2) + d$	$8a + 4b + 2c + d$
3	$a(3)^3 + b(3)^2 + c(3) + d$	$27a + 9b + 3c + d$
4	$a(4)^3 + b(4)^2 + c(4) + d$	$64a + 16b + 4c + d$
5	$a(5)^3 + b(5)^2 + c(5) + d$	$125a + 25b + 5c + d$

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta y = a + b + c$

$\Delta y = 7a + 3b + c$

$\Delta y = 19a + 5b + c$

$\Delta y = 37a + 7b + c$

$\Delta y = 61a + 9b + c$

$6a + 2b$

$12a + 2b$

$18a + 2b$

$24a + 2b$

$6a$

$6a$

$6a$

The third differences are, indeed, constant. Each third difference is $6a$. You can use patterns from the table to determine the quadratic function from the table of data.

- The value of d is the y-coordinate of the y-intercept, $(0, d)$.
- The third difference is equal to $6a$.
- The second difference between the first two pairs of y-values is equal to $6a + 2b$.
- The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

FINITE DIFFERENCES AND CUBIC FUNCTIONS

In a cubic function, the third differences between successive y-values are constant if the differences between successive x-values, Δx, are also constant.

If the third differences between consecutive y-values in a table of values are constant, then the values represent a cubic function.

The formulas for finding the values of a, b, c, and d to write the cubic function only work when $\Delta x = 1$. When $\Delta x \neq 1$, there are other formulas that can be used to determine the values of a, b, c, and d for the cubic function.

EXAMPLE 1

Use the table from the Explore activity to determine the cubic function rule for the volume of a set of rectangular prisms with width = x, length = $2x$, and height = $3x$.

TERM NUMBER, x	PROCESS	VOLUME, y
1	1(2)(3)	6
2	2(4)(6)	48
3	3(6)(9)	162
4	4(8)(12)	384
5	5(10)(15)	750
6	6(12)(18)	1296

STEP 1 Analyze the first, second, and third finite differences to calculate the values of a, b, c, and d for the function rule $f(x) = ax^3 + bx^2 + cx + d$. Use the finite differences to calculate the value of y for $x = 0$.

	TERM NUMBER, x	PROCESS	VOLUME, y			
	0	0(0)(0)	0			
$\Delta x = 1 - 0 = 1$				$\Delta y = 6$		
	1	1(2)(3)	6		36	
$\Delta x = 2 - 1 = 1$				$\Delta y = 42$		36
	2	2(4)(6)	48		72	
$\Delta x = 3 - 2 = 1$				$\Delta y = 114$		36
	3	3(6)(9)	162		108	
$\Delta x = 4 - 3 = 1$				$\Delta y = 222$		36
	4	4(8)(12)	384		144	
$\Delta x = 5 - 4 = 1$				$\Delta y = 366$		36
	5	5(10)(15)	750		180	
$\Delta x = 6 - 5 = 1$				$\Delta y = 546$		
	6	6(12)(18)	1296			

STEP 2 The value of d is the y-coordinate of the y-intercept, (0, d), and for this data set, $d = 0$.

STEP 3 The third difference is equal to $6a$. Since $6a = 36$, $a = 6$.

STEP 4 The second difference between the first two pairs of y-values ($x = 0$ and $x = 1$; $x = 1$ and $x = 2$) is $6a + 2b$.

$$\begin{aligned} 6a + 2b &= 36 \\ 36 + 2b &= 36 \\ 2b &= 0 \\ b &= 0 \end{aligned}$$

STEP 5 The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

$$\begin{aligned} a + b + c &= 6 \\ 6 + 0 + c &= 6 \\ c &= 0 \end{aligned}$$

STEP 6 Write the cubic function rule.

$y = 6x^3 + 0x^2 + 0x + 0$, *or simply* $y = 6x^3$.

YOU TRY IT! #1

Popcorn is sold in boxes in the shape of square prisms. The dimensions of the boxes and their volumes are shown in the table. Write an equation to describe the volume, y, related to the width of the box, x, and verify it with a function rule using the finite differences in the table.

BOX NUMBER	BOX WIDTH IN INCHES, x	WIDTH, LENGTH, AND HEIGHT IN INCHES	VOLUME IN CUBIC INCHES, y
1 (SAMPLER)	1	1(1)(2)	2
2 (KID'S)	2	2(2)(4)	16
3 (SMALL)	3	3(3)(6)	54
4 (MEDIUM)	4	4(4)(8)	128
5 (LARGE)	5	5(5)(10)	250
6 (SUPER)	6	6(6)(12)	432

EXAMPLE 2

Determine whether the data set shown is linear, quadratic, exponential, or cubic.

x	y
-1	34
0	50
1	56
2	58
3	62
4	74
5	100

STEP 1 Determine the first, second, and third differences between successive x-values and successive y-values.

	x	y			
	-1	34			
$\Delta x = 0 - -1 = 1$			$\Delta y = 50 - 34 = 16$		
	0	50		-10	
$\Delta x = 1 - 0 = 1$			$\Delta y = 56 - 50 = 6$		+6
	1	56		-4	
$\Delta x = 2 - 1 = 1$			$\Delta y = 58 - 56 = 2$		+6
	2	58		+2	
$\Delta x = 3 - 2 = 1$			$\Delta y = 62 - 58 = 4$		+6
	3	62		+8	
$\Delta x = 4 - 3 = 1$			$\Delta y = 74 - 62 = 12$		+6
	4	74		+14	
$\Delta x = 5 - 4 = 1$			$\Delta y = 100 - 74 = 26$		
	5	100			

STEP 2 Determine the ratios between successive y-values.

x	y
-1	34
0	50
1	56
2	58
3	62
4	74
5	100

$$\frac{y_n}{y_{n\text{-}1}} = \frac{50}{34} \approx 1.47$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{56}{50} \approx 1.12$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{58}{56} \approx 1.04$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{62}{58} \approx 1.07$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{74}{62} \approx 1.19$$

$$\frac{y_n}{y_{n\text{-}1}} = \frac{100}{74} \approx 1.35$$

STEP 3 Determine whether the set of data represents a linear, quadratic, exponential, or cubic function.

The differences in x, Δx, are all 1, so they are constant.

The first differences are not constant, so the set of data does not represent a linear function.

The second differences are not constant, so the set of data does not represent a quadratic function.

The ratios between successive y-values are not constant, so the set of data does not represent an exponential function.

The third differences are constant, so the set of data represents a cubic function.

YOU TRY IT! #2

Does the set of data shown below represent a cubic function? Justify your answer.

x	y
1	2
2	5
3	11
4	23
5	47
6	99
7	191

EXAMPLE 3

Write a cubic function for the values in the table.

x	y
0	0
1	1
2	2
3	4
4	8
5	15
6	26

STEP 1 **Determine the first differences between successive x-values and the third finite differences in successive y-values.**

x	y
0	0
1	1
2	2
3	4
4	8
5	15
6	26

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

$\Delta y = 1 - 0 = 1$

$\Delta y = 2 - 1 = 1$

$\Delta y = 4 - 2 = 2$

$\Delta y = 8 - 4 = 4$

$\Delta y = 15 - 8 = 7$

$\Delta y = 26 - 15 = 11$

0, 1, 2, 3, 4

1, 1, 1, 1

STEP 2 **Calculate the values for a, b, c, and d in $f(x) = ax^3 + bx^2 + cx + d$.**

- The value of d is the y-coordinate of the y-intercept, $(0, d)$. For this data set, $d = 0$.
- The third difference is equal to $6a$. Since $6a = 1$, $a = \frac{1}{6}$.
- The second difference between the first two pairs of y-values ($x = 0$ and $x = 1$; $x = 1$ and $x = 2$) is $6a + 2b$.

$$6a + 2b = 0$$
$$1 + 2b = 0$$
$$2b = -1$$
$$b = -\frac{1}{2}$$

- The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

$$a + b + c = 1$$
$$\frac{1}{6} - \frac{1}{2} + c = 1$$
$$c = \frac{8}{6} = \frac{4}{3}$$

STEP 3 **Write the cubic function rule with the values of a, b, c, and d:**

$$f(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{4}{3}x.$$

YOU TRY IT! #3

For the data set below, determine if the relationship is a cubic function. If so, write a function relating the variables.

x	y
0	−6
1	$7\frac{1}{2}$
2	18
3	$28\frac{1}{2}$
4	42
5	$61\frac{1}{2}$
6	90

PRACTICE/HOMEWORK

For each table below, determine whether the set of data represents a linear, exponential, quadratic, or cubic function.

1.

x	$f(x)$
-1	0.2
0	1
1	5
2	25
3	125
4	625

2.

x	y
0	-1.25
1	-1
2	-0.75
3	-0.5
4	-0.25
5	0

3.

x	$f(x)$
-1	-5
0	0
1	5
2	40
3	135
4	320

4.

x	$f(x)$
-1	-2
0	-8
1	-2
2	16
3	46
4	88

5.

x	y
-1	8
0	5
1	10
2	29
3	68
4	133

6.

x	y
1	40
2	38
3	36
4	34
5	32
6	30

7. Does the set of data shown below represent a cubic function? Justify your response.

x	y
0	0
1	-4
2	-28
3	-76
4	-148
5	-244

For questions 8 – 10, determine if the given relationship is a cubic function. If it is, write a function relating the variables.

8.

x	y
-1	8
0	5
1	6
2	11
3	20
4	33

9.

x	$f(x)$
0	0
1	0.5
2	4
3	13.5
4	32
5	62.5

10.

x	y
0	-7
1	-5
2	9
3	47
4	121
5	243

For questions 11 – 16, the data sets shown in the tables represent cubic functions. Write a cubic function for the values in the table.

11.

x	y
0	0
1	0.25
2	2
3	6.75
4	16
5	31.25

12.

x	$f(x)$
0	-5
1	-4.8
2	-3.4
3	0.4
4	7.8
5	20

13.

x	y
0	1
1	9
2	57
3	181
4	417
5	801

14.

x	y
0	0
1	-4
2	0
3	18
4	56
5	120

15.

x	$f(x)$
0	-1
1	0.3
2	7.4
3	25.1
4	58.2
5	111.5

16.

x	y
0	0
1	-27
2	-60
3	-63
4	0
5	165

Use the situation below to answer questions 17 – 18.

GEOMETRY

The volume of a set of rectangular prisms with a base length of x inches, is shown below.

LENGTH OF BASE, x (INCHES)	VOLUME, $v(x)$ (CUBIC INCHES)
0	0
1	1.5
2	12
3	40.5
4	96
5	187.5

17. Write the cubic function relating the length of the base to the volume.

18. Use your function to predict the length of the base of the prism when the volume is 1500 cubic inches.

Use the situation below to answer questions 19 – 20.

FINANCE

A local mail service charges different rates, based on the weight of the package being mailed. A sample of their prices is shown in the table below.

WEIGHT OF PACKAGE, w (POUNDS)	PRICE TO MAIL PACKAGE, p ($)
0	0
1	3.45
2	6.60
3	10.65
4	16.80
5	26.25

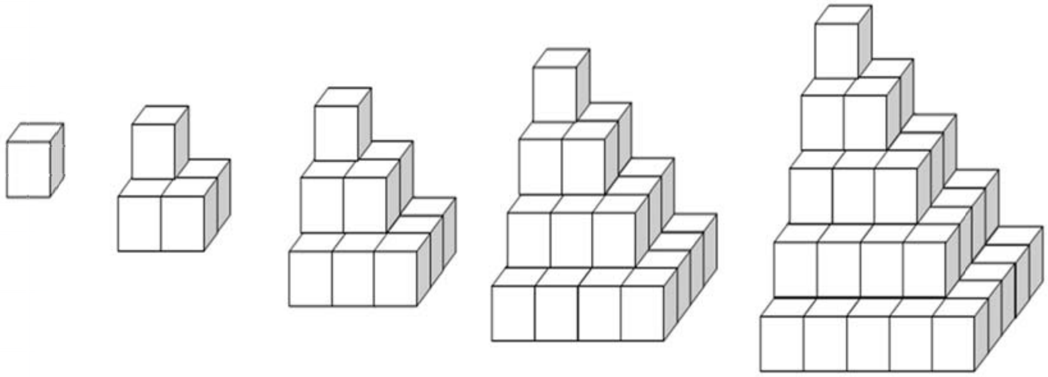
19. Write a cubic function to represent the given data.

20. Use your cubic function to determine the cost to mail a 6-pound package.

Use the situation below to answer questions 21 – 23.

CRITICAL THINKING

Blocks were stacked to create the pattern below.



21. Relate the number of layers in a stack, x, to the total number of blocks, y, by completing the table below. The first few rows have been completed for you.

NUMBER OF LAYERS, x	TOTAL NUMBER OF BLOCKS, p
0	0
1	1
2	5
3	14
4	
5	

22. Write the function relating the variables in problem 21.

23. If the pattern continues, how many blocks would it take to create a 7-layer stack?

Modeling with Cubic Functions

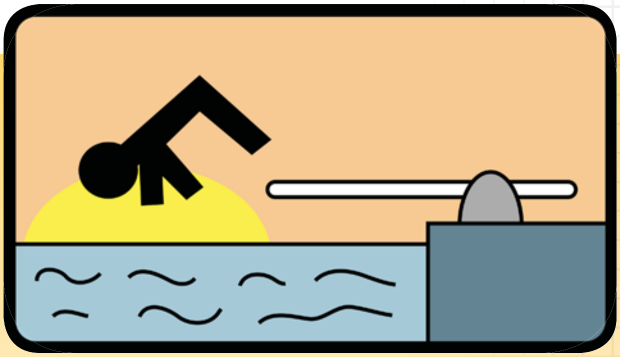
1.9

FOCUSING QUESTION How can you use finite differences to construct a cubic model for a data set?

LEARNING OUTCOMES

- I can use finite differences or common ratios to classify a function as linear, quadratic, cubic, or exponential when I am given a table of values.
- I can determine the cubic function from a table using finite differences, including any restrictions on the domain and range.
- I can use finite differences to write a cubic function that describes a data set.
- I can apply mathematics to problems that I see in everyday life, in society, and in the workplace.

ENGAGE



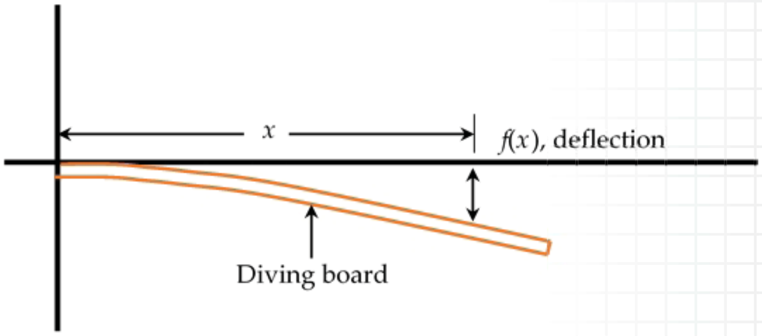
A diving board is a platform that someone can use to dive into a deep water pool. Diving boards typically have enough flexibility to allow a diver to bounce before leaving the board and entering the pool. Diving boards have a heel end on the pool deck, which is where the person steps onto the diving board, and a toe end that hangs over the pool. Diving boards also have a fulcrum that balances the board.

When a person stands on a diving board, it bends or deflects slightly due to the person's weight. What questions could you ask about the deflection of a diving board that could be answered by collecting data?

Work with a partner to give information you know about diving boards and this situation.

EXPLORE

When a person stands on a diving board, the diving board bends beneath the weight of the person. The farther the person stands from the fulcrum of the diving board, the more the diving board bends, or deflects, from the horizontal.



The table below shows the amount of deflection, in thousandths of an inch, when the same person stands x feet from the fulcrum of the diving board.

DISTANCE FROM FULCRUM (FT), x	DEFLECTION (0.001 IN.), $f(x)$
0	0
1	116
2	448
3	972
4	1664
5	2500

1. Does the data set appear to have a constant rate of change in deflection? Explain how you know.

2. Does the amount of deflection appear to increase by the same factor each time the distance increases by 1 foot? Explain how you know.

3. Calculate the finite differences between the deflection and the distance from the fulcrum. Do the data appear to be linear, quadratic, or cubic? How do you know?

4. Use the patterns in the finite differences to write a function rule that describes the data set.

5. Use a graphing calculator to graph the function rule over a scatterplot of the data. What do you notice about the function rule and the scatterplot?

6. Compare the domain and range of the data set and the domain and range of the function rule. How are they alike? How are they different?

7. What will be the deflection if the person stands 10 feet from the fulcrum?

8. Are the x-intercepts of the function included in your data set? Why or why not?

9. What dimension of the diving board represents the greatest possible x-value that could be contained in the data set? What limit does that place on the domain of the data set?

REFLECT

- **How can you determine a cubic function model for a data set?**
- **What other factors could influence the deflection of the diving board? Explain how they would do so.**

EXPLAIN

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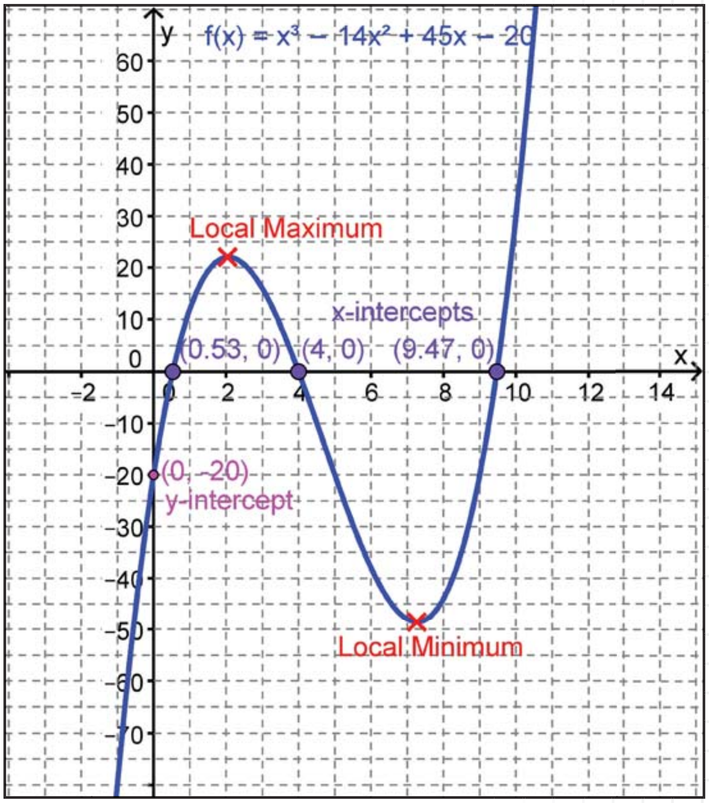
Cubic function models can be used to represent sets of mathematical and real-world data. Cubic functions have several attributes that should be considered when they are used for mathematical models.

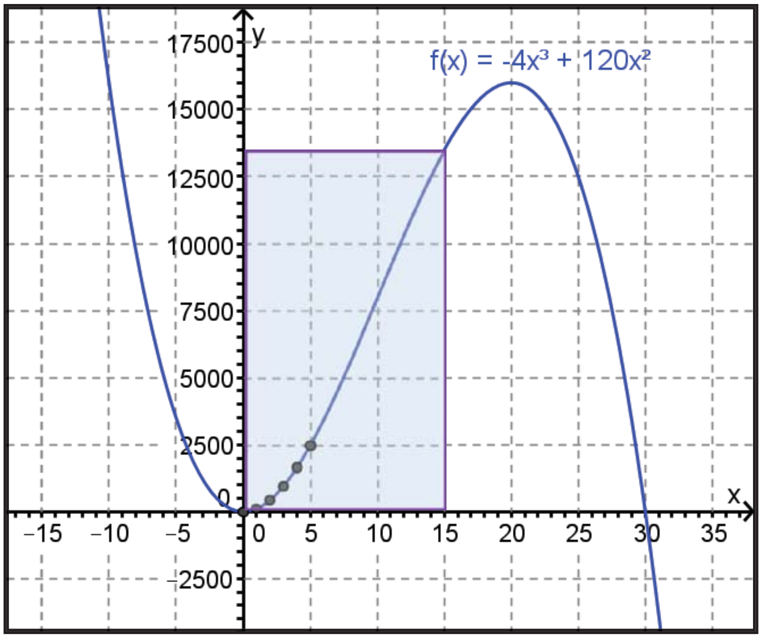
The domain and range of most cubic functions are all real numbers. Unlike quadratic functions, when you raise a negative number to the third power, you can have a negative number as a result.

Cubic functions have as many as 3 x-intercepts and 1 y-intercept. As with other function types, the x-intercepts represent x-values that generate a function value equal to 0. The y-intercept is also a starting point, or y-value when $x = 0$.



Quadratic functions have a maximum or minimum point at the vertex. These are absolute maximum or minimum values. A cubic function, however, may have what are called **local maximum** or **local minimum** values. These points are not the absolute maximum or minimum values for the function. Instead, they are maximum or minimum values for a nearby, or local, part of the function.

Let's look more closely at the data set and function model for the diving board problem.



In the diving board problem, the data points represent a small portion of the function. The shaded region on the graph shows the part of the function that would model the deflection of a diving board that has a distance of 15 feet between the end of the diving board and the fulcrum.

The function has two *x*-intercepts, one of which makes sense in the context of the diving board problem. When the person stands 0 feet from the fulcrum, you would expect the diving board to deflect 0 inches. However, it is not likely that a person standing 30 feet from the fulcrum would generate a deflection of 0 inches.

With **polynomial function models**, frequently the function model only represents the data set for a certain interval of the function. In the diving board problem, there is a local maximum at (20, 16,000). This means that a person standing 20 feet from the fulcrum causes a deflection of 16.000 inches. The function model for *x*-values greater than 20 then begins to decrease. It is not likely that a person standing farther than 20 feet from the fulcrum would generate less deflection, so the model likely does not represent the situation beyond a domain of 20 feet.

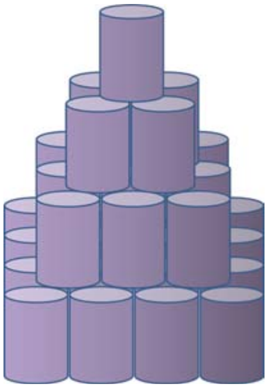
MODELING WITH CUBIC FUNCTIONS

If the data set has a constant or approximately constant third finite difference, then a cubic function model may be appropriate for the data set.

The domain and range for the situation may be a subset of the domain and range of the cubic function model. Cubic functions can have intervals within their domain when they are increasing and intervals within their domain when they are decreasing.

EXAMPLE 1

A display of large juice cans takes the shape of a square-based pyramid with 1 can in the top level, 4 cans in the second level, 9 in the third level, 16 in the fourth level, etc. To predict the number of cans y for a display of x number of levels, determine if this situation represents a linear, quadratic, or cubic function.



LEVELS, x	**PROCESS**	**NUMBER OF CANS, y**
1	1	1
2	1 + 4	5
3	1 + 4 + 9	14
4	1 + 4 + 9 + 16	30
5	1 + 4 + 9 + 16 + 25	55
6	1 + 4 + 9 + 16 + 25 + 36	91

Use the data set to determine if the relationship is linear, quadratic, or cubic.

STEP 1 Consider the value of y, the number of cans, in the “zero” level of the display. The “zero” level would logically have no cans.

STEP 2 Determine the finite differences in values of x and the first, second, and third finite differences in values of $f(x)$.

LEVELS, x	**PROCESS**	**NUMBER OF CANS, y**
0	0	0
1	1	1
2	1 + 4	5
3	1 + 4 + 9	14
4	1 + 4 + 9 + 16	30
5	1 + 4 + 9 + 16 + 25	55
6	1 + 4 + 9 + 16 + 25 + 36	91

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

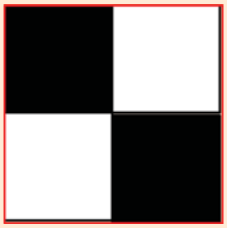
$+1$, $+4$, $+9$, $+16$, $+25$, $+36$

$+3$, $+5$, $+7$, $+9$, $+11$

$+2$, $+2$, $+2$, $+2$

The set of data represents a cubic function because the differences in x are constant and the third finite differences between successive values of f(x) are constant.

YOU TRY IT! #1



The number of squares on a checkerboard, including the number of 1 × 1 squares, 2 × 2 squares, 3 × 3 squares, and so on, are shown in the table. If the checkerboard is 1 × 1, there is only one square possible. In a 2 × 2 board, there are four 1 × 1 squares and one 4 × 4 square, for a total of five squares. The chart below shows the total number of squares contained in ***two*** checkerboards.

SIDE LENGTH OF BOARD IN SQUARES, x	TOTAL NUMBER OF SQUARES, $f(x)$
1	2
2	10
3	28
4	60
5	110
6	182
7	280
8	408

Determine whether the relationship is linear, quadratic, or cubic.

EXAMPLE 2

Determine the cubic function rule to model the display of cans in Example 1. Compare the domain and range of the data set and the function.

Step 1 Calculate the values of a, b, c, and d in the cubic function rule $f(x) = ax^3 + bx^2 + cx + d$ using the data in the table and finite differences.

- There would be no cans in the "zero" level. The value for y when $x = 0$ is 0. So $d = 0$.
- The third finite difference in the set of data is 2. This equals $6a$. If $6a = 2$, then $a = \frac{1}{3}$.

- The second difference between the first two pairs of y-values ($x = 0$ and $x = 1$; $x = 1$ and $x = 2$) is equal to $6a + 2b$.

$$6a + 2b = 3$$
$$2 + 2b = 3$$

$$2b = 1 \text{ and } b = \frac{1}{2}$$

- The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

$$a + b + c = 1$$
$$\frac{1}{3} + \frac{1}{2} + c = 1$$
$$c = \frac{1}{6}$$

STEP 3 **Write the cubic function rule with the values of a, b, c, and d:**

$$f(x) = \frac{1}{2}x^3 + \frac{1}{3}x^2 + \frac{1}{6}x$$

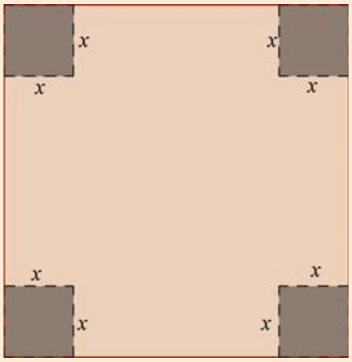
STEP 4 **Enter the x- and y-values from the table and then graph the function with technology. The graph connects the data points. The data points are limited to values for levels 1 to 8.**

- Considering the weight of the cans in several levels, it doesn't make sense to add more levels. Also, the function rule produces fractional parts of cans for many values of x.
- Data set - domain: whole numbers, $1 \le x \le 8$; range: {1, 5, 14, 30, 55, 91, 140, 204}
- Function - domain: all real numbers; range: all real numbers

The domain and range of the data set are subsets of the domain and range of the function rule.

The domain and range of the data set are limited to whole numbers, but the domain and range of the function rule include all real numbers.

YOU TRY IT! #2



A tray is made of a 12 in. **x** 12 in. square piece of cardboard with squares cut out of the four corners. The resulting sides are folded up and taped to form a square prism (open box). The volume of the tray related to the size of the squares cut from the corners.

SIZE OF SQUARE CUTS IN INCHES, x	PROCESS $(12 - 2x)(12 - 2x)(x)$	VOLUME IN CUBIC INCHES, $f(x)$
0	(12 – 0)(12 – 0)(0)	0
1	(12 – 2)(12 – 2)(1)	100
2	(12 – 4)(12 – 4)(2)	128
3	(12 – 6)(12 – 6)(3)	108
4	(12 – 8)(12 – 8)(4)	64
5	(12 – 10)(12 – 10)(5)	20
6	(12 – 12)(12 – 12)(6)	0

Generate a cubic function model for the volume of the tray. What size cutout produces the tray with the greatest volume? Identify the domain and range that most appropriately models the data.

EXAMPLE 3

A set of graduated cubes comes with a large industrial scale. The cubes are numbered according to increasing size. The weights of some of the cubes are given in the table. Determine if the data set represents a linear, quadratic, or cubic function. Identify the domain and range of the model that most appropriately models the data.

CUBE NUMBER, x	WEIGHT IN OUNCES, $f(x)$
2	12.5
4	34.3
6	72.9
8	133.1
10	219.7
12	337.5

STEP 1 Determine the finite differences in values of x and the first, second, and third finite differences in values of $f(x)$.

	CUBE NUMBER, x	WEIGHT IN OUNCES, $f(x)$			
	2	12.5			
$\Delta x = 4 - 2 = 2$			+21.8		
	4	34.3		+16.8	
$\Delta x = 6 - 4 = 2$			+38.6		+4.8
	6	72.9		+21.6	
$\Delta x = 8 - 6 = 2$			+60.2		+4.8
	8	133.1		+26.4	
$\Delta x = 10 - 8 = 2$			+86.6		+4.8
	10	219.7		+31.2	
$\Delta x = 12 - 10 = 2$			+117.8		
	12	337.5			

The set of data represents a cubic function because the differences in x are constant, although the Δx is 2, not 1, and the third finite differences between successive values of $f(x)$ are constant.

STEP 2 From the information, it is unclear if there are larger or odd-numbered cubes.

The set of data represents a cubic function.
The domain is a subset of whole numbers, $0 < x \leq 12$.
The range is a subset of real numbers, $0 < y \leq 337.5$.

YOU TRY IT! #3

The power, y, (in kilowatts) generated by a wind turbine is related to the wind speed. Determine if the data set represents a linear, quadratic, or cubic function. Identify the domain and range of the model that most appropriately models the data.

AVERAGE ANNUAL WIND SPEED IN THE USA IN METERS PER SECOND, x	ANNUAL ENERGY OUTPUT IN KWH/YEAR, $f(x)$
4	3.40
5	6.64
6	11.47
7	18.22
8	27.25
9	38.75
10	53.12

Data Source: National Renewal Energy Laboratory and Energy.gov

PRACTICE/HOMEWORK

For questions 1 – 6, use finite differences to determine if the data sets shown in the tables below represent a linear, exponential, quadratic, or cubic function.

1.

x	y
0	3
1	6
2	12
3	24
4	48
5	96

2.

x	y
0	-6
1	1
2	16
3	39
4	70
5	109

3.

x	y
0	2.25
1	8.75
2	15.25
3	21.75
4	28.25
5	34.75

4.

x	y
0	5
1	10
2	35
3	92
4	193
5	350

5.

x	y
0	20
1	50
2	125
3	312.5
4	781.25
5	1953.125

6.

x	y
0	-8
1	3
2	44
3	145
4	336
5	647

For questions 7 – 12, the data sets shown in the tables represent cubic functions. Use finite differences to determine the function that relates the variables.

7.

x	y
0	-1
1	0
2	11
3	50
4	135
5	284

8.

x	y
0	3
1	7
2	1
3	-27
4	-89
5	-197

9.

x	y
0	0
1	14
2	72
3	198
4	416
5	750

10.

x	y
0	-4
1	-9
2	-48
3	-157
4	-372
5	-729

11.

x	y
0	1
1	6
2	15
3	31
4	57
5	96

12.

x	y
0	-9
1	-3
2	39
3	153
4	375
5	741

For questions 13 – 17 use the scenario below.

GEOMETRY

A box is created from a 20-inch by 24-inch rectangular piece of cardboard by cutting congruent squares from each corner. The squares are cut in 1-inch increments. The resulting sides are folded up and taped to form a rectangular prism (open box). The volume of the box is a function of the side length of the square removed from each corner. The table below relates the volume of the box to the side length of the square.

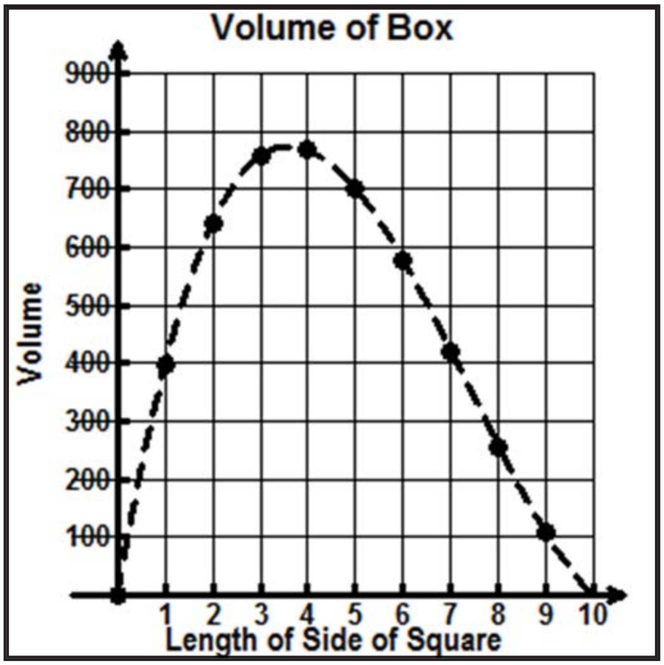
SIDE LENGTH x	VOLUME y
0	0
1	396
2	640
3	756
4	768
5	700
6	576
7	420
8	256
9	108

13. Generate a cubic function model for the volume of tray when given the side length of the square cut from each of the corners.

14. What side length of the square produces a tray with the greatest volume?

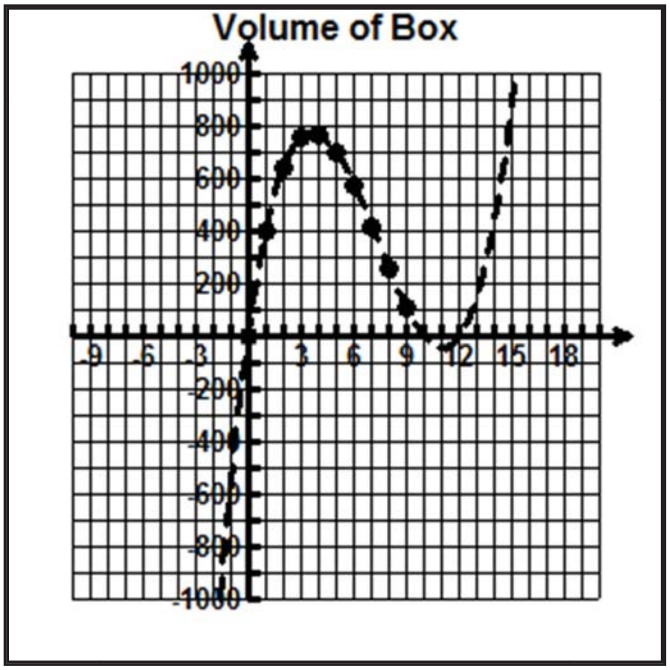
15. The graph to the right represents the data in the table and the function that models the table.

What is the domain and range of this <u>situation</u>?



16. The graph to the right represents the data in the table and the function that models the table, but has been graphed with a different window setting.

Why are there no individual points plotted on the graph for $x < 0$ and $x > 10$?



17. Why does the domain and range only contain whole numbers?

For questions 18 – 22 use the scenario below.

CRITICAL THINKING

An employee at a toy store is creating a display of soccer balls in the shape of a tetrahedron, or an equilateral triangular pyramid.

The table below shows the total number of soccer balls at each level of the display, with Level 1 being at the top of the display.

LEVEL, x	**TOTAL NUMBER OF SOCCER BALLS, y**
1	1
2	4
3	10
4	20
5	35
6	56

18. Write a function using finite differences that models the data in the table.

19. What does the domain of the function represent in the situation?

20. What does the range of the function represent in the situation?

21. Is 2.5 an element in the domain of this situation? Why or why not?

22. How many soccer balls would be needed to build a display 10 levels high?

Chapter 1 Review

1. Given each rule, write the first 4 terms of the sequence. Indicate whether each sequence is arithmetic or geometric.

a) $a_n = 256\left(\frac{3}{4}\right)^{n-1}$

c) $a1 = \text{-}54;\ a_n = a_{n-1} + 7$

b) $a_1 = 2;\ a_n = 3a_{n-1}$

d) $a_n = 3250 - 75n$

2. Write an equation in $y = mx + b$ form for each linear function described.

a) slope = $\frac{5}{2}$, y-intercept = (0, 10)

b) slope = −2, contains the point (5, −7)

c) contains the points (5, -3) and (-4, 0)

d)

x	y
2	21
4	18
6	15
8	12
10	9

3. For each table below representing exponential data, write the common ratio and the function relating the variables.

a)

x	y
0	243
1	81
2	27
3	9

b)

x	y
1	500
2	1000
3	2000
4	4000

4. The population of a certain bacteria over a period of time is shown in the table below, where x represents the number of days since the bacteria's population was first recorded and y represents the population.

NUMBER OF DAYS, x	0	1	2	3	4	5	6	7
POPULATION, y	123	290	715	1,695	4,048	9,769	23,500	56,400

a. Is the data, linear, quadratic, cubic, or exponential?
b. Generate a function to model this data.
c. According to your model, what will be the population of the bacteria after 9 days?
d. According to your model, when will the population of the bacteria exceed 10,000,000?

For problems 5 – 10, determine if the function represented is linear, quadratic, cubic, or exponential, then write a function equation relating the variables.

5.

x	y
1	11
2	22
3	37
4	56
5	79

6.

x	y
0	12
2	9
4	6
6	3
8	0

7.

x	y
0	8
1	12
2	18
3	27
4	40.5

8.

x	y
0	-8
1	-6
2	8
3	46
4	120

9.

x	y
0	6
1	$9\frac{1}{3}$
2	$26\frac{2}{3}$
3	60
4	$111\frac{1}{3}$

10.

x	y
-2	19
-1	13
0	5
1	-5
2	-17

Use the following situation to answer questions 11-14.

Amanda drew a sequence of figures as shown. She then recorded the number of dots used for each figure and recorded the data in a table.

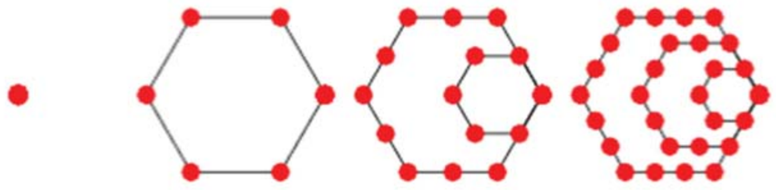
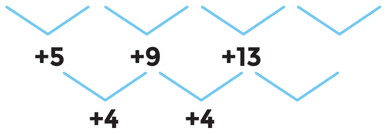


FIGURE NUMBER, n	1	2	3	4	5	n
NUMBER OF DOTS, $D(n)$	1	6	15	28		

11. Using the pattern of second differences, determine the number of dots that would appear in the 5th figure.

FIGURE NUMBER, n	1	2	3	4	5	n
NUMBER OF DOTS, $D(n)$	1	6	15	28		



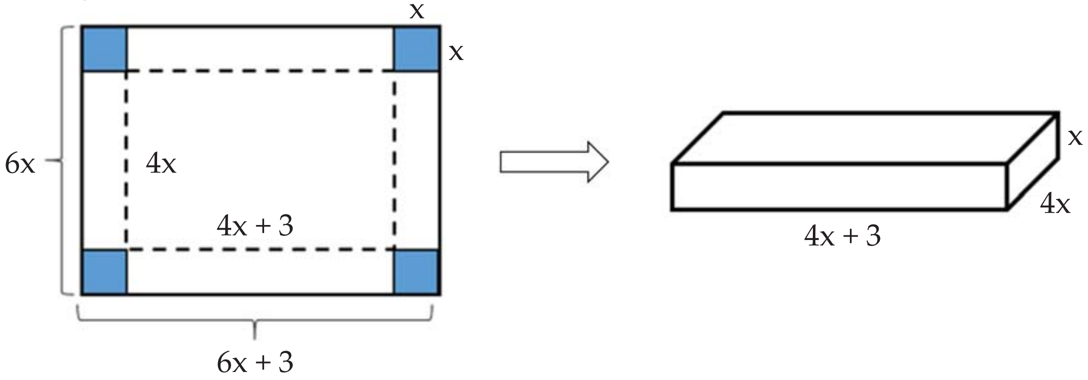
12. Write a quadratic function to represent the relationship between n and $D(n)$.

13. How many dots would appear in the 9th figure?

14. In which figure would there be 435 dots?

Use the following situation to answer questions 15-17.

Fred created a box out of a sheet of cardboard in which the length was 3 inches longer than its width. The height of the box was created by cutting squares from each corner of the cardboard and folding up $\frac{1}{6}$ of the width of the cardboard on all sides as shown in the figure.

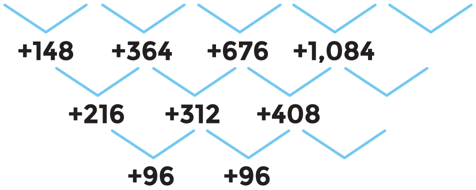


Fred made several boxes that were similar in dimensions to the first box and recorded the volume of each as compared to the height.

HEIGHT OF BOX (INCHES), x	1	2	3	4	5	x
VOLUME OF BOX (CUBIC INCHES), V	28	176	540	1,216	2,300	

15. Using the pattern of third differences, determine the volume of a box with a height of 6 inches.

HEIGHT OF BOX (INCHES), x	1	2	3	4	5	x
VOLUME OF BOX (CUBIC INCHES), V	28	176	540	1,216	2,300	



16. Write a cubic function to represent the relationship between x and V.

17. What would be the volume of a box with a height of 2.5 inches?

Use the following situation to answer questions 18-20.

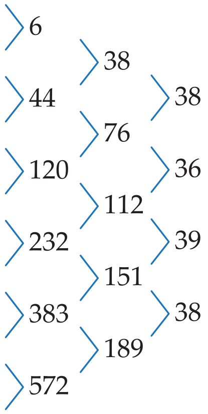
A shipping company packs candied popcorn in cylindrical cartons in which the height and radius are the same. Various sizes of the filled cartons and weights are shown in the table below.

RADIUS OF CARTON, r (IN)	WEIGHT OF CARTON, W (OUNCES)
1	6
2	50
3	170
4	402
5	785
6	1357

18. Does the data best fit a linear, quadratic, cubic, or exponential model? Justify your answer.

19. Write a function to model the relationship between r and W.

RADIUS OF CARTON, r (IN)	WEIGHT OF CARTON, W (OUNCES)
0	0
1	6
2	50
3	170
4	402
5	785
6	1357



20. According to your model, what would be the approximate height of a carton that weighed about 138 ounces?

MULTIPLE CHOICE

21. Rhonda has \$150 in her savings account but plans to add \$30 each week from money she earns from babysitting. Which function represents the balance of her savings account, S, after w weeks?

A. $S = 30 + 150w$
B. $S = 150(30)w$
C. $S = (150 + 30)w$
D. $S = 150 + 30w$

22. In a table showing values of x and $f(x)$ from the quadratic function, $f(x) = ax^2 + bx + c$, the second differences of the f(x) values gives you what information related to the function?

A. a
B. 2a
C. 6a
D. a + b + c

23. Which of the following equations best models the quadratic data given below?

x	0	1	2	3	4	5	6
y	15	19	35	63	103	155	219

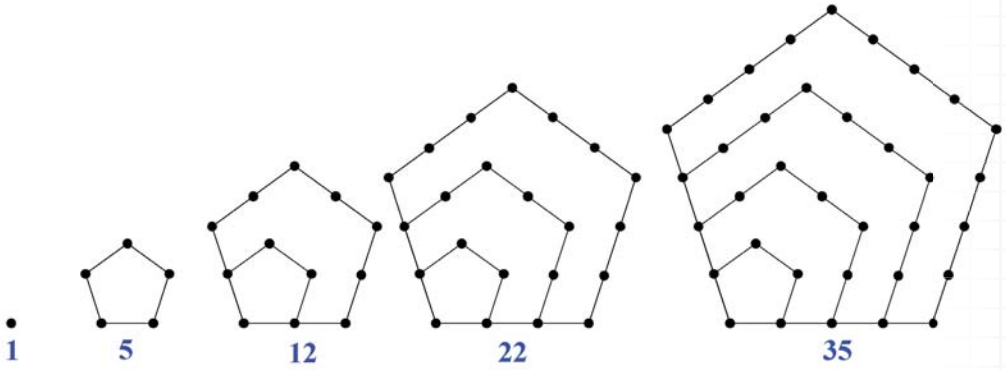
A. $6x^2 - 1.9x + 14.8$
B. $6x^2 + 1.9x + 14.8$
C. $x^2 - 1.9x + 4.2$
D. $3x^2 - 19x + 18.4$

24. For the data shown in the table below, what is the coefficient of the x^3 term?

x	0	1	2	3	4	5	6	7
$y = ax^3 + bx^2 + cx + d$	-200	-186	-144	-50	120	390	784	1326

A. 24
B. 12
C. 6
D. 4

25. In a sequence of pentagonal numbers, the n^{th} figure consists of $\frac{3n^2 - n}{2}$ dots.



How many dots make up the 9th pentagonal figure?

A. 92
B. 117
C. 234
D. 360